Exercises on metric and Hilbert spaces
An invitation to functional analysis

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1 Introduction

The next few exercises are about countability/uncountability. See Section 1.2 for clarification on our use of the term "countable". You may assume without proof that any subset of a countable set is finite or countable.

Exercise 1.1. Let $f\colon X \longrightarrow Y$ be a function, with X a countable set. Then $\mathrm{im}(f)$ is finite or countable.

[*Hint*: Reduce to the case $f\colon \mathbf{N} \longrightarrow Y$ is surjective; construct a right inverse $g\colon Y \longrightarrow \mathbf{N}$, which has to be injective, of f.]

Solution. Without loss of generality, we may assume that f is surjective and we want to show that Y is finite or countable.

Also without loss of generality (by pre-composing f with any bijection $\mathbf{N} \longrightarrow X$), we may assume that $f\colon \mathbf{N} \longrightarrow Y$ is surjective.

As $f\colon \mathbf{N} \longrightarrow Y$ is surjective, there exists a right inverse $g\colon Y \longrightarrow \mathbf{N}$, in other words $f \circ g\colon Y \longrightarrow Y$ is the identity function id_Y: given $y \in Y$, the pre-image $f^{-1}(y) \subseteq \mathbf{N}$ is nonempty, so it has a smallest element n_y; we let $g(y) = n_y$. For any $y \in Y$, we have $f(g(y)) = f(n_y) = y$ as $n_y \in f^{-1}(y)$. So $f \circ g = \mathrm{id}_Y$.

In particular, this forces $g\colon Y \longrightarrow \mathbf{N}$ to be injective, hence realising Y as a subset of the countable set $\mathbf{N}$. We conclude that Y is finite or countable. □

Exercise 1.2. Show that the union S of any countable collection of countable sets is a countable set.

[*Hint*: Construct a surjective function $\mathbf{N} \times \mathbf{N} \longrightarrow S$.]

Solution. Write

$$S = \bigcup_{n \in \mathbf{N}} S_n,$$

with each S_n a countable set. It is clear that S is infinite (as, say, S_1 is, and $S_1 \subseteq S$).

For each $n \in \mathbf{N}$, fix a bijection $\varphi_n\colon \mathbf{N} \longrightarrow S_n$. (As Chengjing rightfully points out to me, this uses the Axiom of Countable Choice.) Define a function $\psi\colon \mathbf{N} \times \mathbf{N} \longrightarrow S$ by:

$$\psi\big((n, m)\big) = \varphi_n(m) \in S_n \subseteq S.$$

This is surjective, and $\mathbf{N} \times \mathbf{N}$ is countable, so S is finite or countable, and we ruled out finite above. □

Exercise 1.3. Let $\mathbf{R}^\infty$ be the set of arbitrary sequences $(x_1, x_2, \dots)$ of elements of $\mathbf{R}$.

This is a vector space under the naturally-defined addition of sequences and multiplication by a scalar.

Let $e_j \in \mathbf{R}^\infty$ be the sequence whose j-th entry is 1, and all the others are 0. Describe the subspace $\mathrm{Span}\{e_1, e_2, \dots\}$ of $\mathbf{R}^\infty$. Is the set $\{e_1, e_2, \dots\}$ a basis of $\mathbf{R}^\infty$?

Solution. Let $S = \{e_1, e_2, \dots\}$ and $W = \mathrm{Span}(S)$.
For each $n \in \mathbf{N}$, define

$$W_n = \mathrm{Span}\{e_1, e_2, \dots, e_n\} \subseteq W.$$

I claim that

$$W = \bigcup_{n \in \mathbf{N}} W_n.$$

One inclusion is clear, as $W_n \subseteq W$ for all $n \in \mathbf{N}$.

For the other inclusion, let $w \in W$. Then there exist $m \in \mathbf{N}$, $a_1, \dots, a_m \in \mathbf{R}$ and $k_1, \dots, k_m \in \mathbf{N}$ such that

$$w = a_1 e_{k_1} + \cdots + a_m e_{k_m}.$$

Set $n = \max\{k_1, \dots, k_m\}$, then $w \in W_n$.

Is $W = \mathbf{R}^\infty$? No. Any $w \in W$ appears in a W_n for some $n \in \mathbf{N}$, therefore only the first n entries of w can be nonzero. This means, for instance, that $v = (1, 1, 1, \dots) \notin W$. So S does not span $\mathbf{R}^\infty$. □

Exercise 1.4. Let $V = \mathbf{R}$ viewed as a vector space over $\mathbf{Q}$.

Let $\alpha \in \mathbf{R}$. Show that the set $T = \{\alpha^n : n \in \mathbf{N}\}$ is $\mathbf{Q}$-linearly independent if and only if α is transcendental.

(Note: An element $\alpha \in \mathbf{R}$ is called *algebraic* if there exists a monic polynomial $f \in \mathbf{Q}[x]$ such that $f(\alpha) = 0$. An element $\alpha \in \mathbf{R}$ is called *transcendental* if it is not algebraic.)

Solution. This is a straightforward rewriting of the definition of algebraic: α is algebraic if and only if it satisfies a polynomial equation with coefficients in $\mathbf{Q}$, which is equivalent to a nontrivial linear relation between the powers of α, which exists if and only if T is linearly dependent. □

Exercise 1.5. Let W be a $\mathbf{Q}$-vector space with a countable basis B. Show that W is a countable set.

[*Hint*: Use Exercise 1.2.]

Conclude that $\mathbf{R}$ does not have a countable basis as a vector space over $\mathbf{Q}$.

Solution. Since B is countable we can enumerate it as $B = \{b_n : n \in \mathbf{N}\}$. For each $n \in \mathbf{N}$, let $W_n = \mathrm{Span}\{b_1, \dots, b_n\}$. Then for each $n \in \mathbf{N}$, W_n is isomorphic (as a $\mathbf{Q}$-vector space) to $\mathbf{Q}^n$, hence W_n is countable. I claim that

$$W = \bigcup_{n \in \mathbf{N}} W_n.$$

One inclusion is obvious, as $W_n \subseteq W$ for all $n \in \mathbf{N}$. For the other direction, let $w \in W = \mathrm{Span}(B)$, so there exist $n \in \mathbf{N}$, $a_1, \dots, a_n \in \mathbf{Q}$ and $k_1, \dots, k_n \in \mathbf{N}$ such that

$$w = a_1 b_{k_1} + \cdots + a_n b_{k_n}.$$

Let $k = \max\{k_1, \dots, k_n\}$, then $w \in W_k$.

So W is a countable union of countable sets, hence countable by Exercise 1.2.

The last claim follows directly from the fact that $\mathbf{R}$ is an uncountable set. □

We now turn to posets, Zorn's Lemma, and the existence of bases.

A **p**artially **o**rdered **set** (*poset* for short) is a set X together with a *partial order* $\leqslant$, that is a relation satisfying

- $x \leqslant x$ for all $x \in X$;
- if $x \leqslant y$ and $y \leqslant x$ then $x = y$;
- if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$.

A poset X such that for any $x, y \in X$ we have $x \leqslant y$ or $y \leqslant x$ is called a *totally ordered set*, and $\leqslant$ is called a *total order*.

Exercise 1.6. Fix a set Ω and let X be the set of all subsets of Ω. Check that $\subseteq$ is a partial order on X. It is not a total order if Ω has at least two distinct elements.

Solution. The fact that $\subseteq$ is a partial order follows directly from known properties of set inclusion.

If Ω has at least two distinct elements x_1 and x_2, then $\{x_1\}$ and $\{x_2\}$ are not comparable under $\subseteq$, so the latter is not a total order. □

A *chain* in a poset $(X, \leqslant)$ is a subset $C \subseteq X$ that is totally ordered with respect to $\leqslant$.

If $S \subseteq X$ is a subset of a poset, then an *upper bound* for S is an element $u \in X$ such that $s \leqslant u$ for all $s \in S$.

A *maximal element* of a poset X is an element m of X such that there does not exist any $x \in X$ such that $x \neq m$ and $m \leqslant x$. In other words, for any $x \in X$, either $x = m$, or $x \leqslant m$, or x and m are not comparable with respect to the partial order $\leqslant$.

Exercise 1.7. Let $(X, \leqslant)$ be a nonempty finite poset. (This just means that X is a nonempty finite set with a partial order $\leqslant$.) Prove that X has a maximal element.

[*Hint*: You could, for instance, use induction on the number of elements of X.]

Solution. We proceed by induction on n, the cardinality of X.

Base case: if $n = 1$ then $X = \{x\}$ for a single element x. Then trivially x is a maximal element of X.

For the induction step, fix $n \in \mathbf{N}$ and suppose that any poset of cardinality n has a maximal element. Let X be a poset of cardinality $n + 1$ and choose an arbitrary element $x \in X$. Let $Y = X \setminus \{x\}$, then Y is a poset of cardinality n so by the induction hypothesis has a maximal element m_Y, and clearly $m_Y \neq x$.

We have two possibilities now:

- If $m_Y \leqslant x$, then x is a maximal element of X. Why? Suppose that x is not maximal in X, so that there exists $z \in X$ such that $z \neq x$ and $x \leqslant z$. Since $z \neq x$, we must have $z \in Y$. If $z = m_Y$, then $z \leqslant x$ and $x \leqslant z$ so $z = x$, contradiction. So $z \neq m_Y$, and $m_Y \leqslant x$ and $x \leqslant z$, so $m_Y \leqslant z$, contradicting the maximality of m_Y in Y.
- Otherwise, (if it is not true that $m_Y \leqslant x$), m_Y is a maximal element of X. Why? Suppose there exists $z \in X$ such that $z \neq m_Y$ and $m_Y \leqslant z$. Since $m_Y \leqslant x$ is not true, we have $z \neq x$, so $z \in Y$, contradicting the maximality of m_Y in Y.

In either case we found a maximal element for X.

An alternative approach is to proceed by contradiction: suppose $(X, \leqslant)$ is a nonempty finite poset that does not have a maximal element. Use this to construct an unbounded chain of elements of X, contradicting finiteness. □

Zorn's Lemma (Lemma 1.3) is used to deduce the existence of maximal elements in infnite posets.

Exercise 1.8. Prove Theorem 1.2: any vector space V has a basis.

[*Hint*: Let X be the set of all linearly independent subsets of V, partially ordered by inclusion. Prove that X has a maximal element B, and prove that this must also span V.]

Solution. If $V = \{0\}$, then $\varnothing$ is vacuously a (in fact, the only) basis of V.

Suppose $V \neq \{0\}$. If $v \in V \setminus \{0\}$, then $\{v\}$ is a linearly independent subset of V. Let X be the set of all linearly independent subsets of V, then X is nonempty. We consider the partial order $\subseteq$ on X given by inclusion of subsets.

Let C be a nonempty chain in X and define

$$U = \bigcup_{S \in C} S,$$

then clearly $S \subseteq U$ for all $S \in C$, so we'll know that U is an upper bound for C as soon as we can show that it is linearly independent (so that $U \in X$).

Suppose there exist $n \in \mathbf{N}$, $a_1, \dots, a_n \in \mathbf{F}$, and $u_1, \dots, u_n \in U$ such that

$$a_1 u_1 + \cdots + a_n u_n = 0. \tag{1.1}$$

Let $J = \{1, \dots, n\}$. For each $j \in J$, there exists $S_j \in C$ such that $u_j \in S_j$. As C is totally ordered, there exists $i \in J$ such that $S_j \subseteq S_i$ for all $j \in J$. But this means that $u_1, \dots, u_n \in S_i$, so that the linear relation of Equation (1.1) takes place in the linearly independent set S_i. Therefore $a_1 = \cdots = a_n = 0$.

We conclude that X satisfies the conditions of Zorn's Lemma, hence it has a maximal element B. I claim that B spans V, so that it is a basis of V.

We prove this last claim by contradiction: if $v \in V \setminus \operatorname{Span}(B)$, then $B' := B \cup \{v\}$ is linearly independent, hence an element of X. But $B \subseteq B'$ and $B \neq B'$, contradicting the maximality of B. □

2 Metric and topological spaces

Exercise 2.1. Let (X, d) be a metric space. Show that

$$|d(x,y) - d(t,y)| \leqslant d(x,t)$$

for all $x, y, t \in X$.

Solution. We need to show that

$$-d(x,t) \leqslant d(x,y) - d(t,y) \leqslant d(x,t).$$

One application of the triangle inequality gives

$$d(x,y) \leqslant d(x,t) + d(t,y) \qquad \Rightarrow \qquad d(x,y) - d(t,y) \leqslant d(x,t).$$

Another application gives

$$d(t,y) \leqslant d(t,x) + d(x,y) \qquad \Rightarrow \qquad -d(x,t) \leqslant d(x,y) - d(t,y).$$ □

Exercise 2.2. Let (X, d) be a metric space. Show that

$$|d(x,y) - d(s,t)| \leqslant d(x,s) + d(y,t)$$

for all $x, s, y, t \in X$.

Solution. We have

$$\begin{aligned} |d(x,y) - d(s,t)| &= |d(x,y) - d(y,s) + d(y,s) - d(s,t)| \\ &\leqslant |d(x,y) - d(y,s)| + |d(y,s) - d(s,t)| \\ &\leqslant d(x,s) + d(y,t) \end{aligned}$$

after one application of the triangle inequality and two applications of Exercise 2.1. □

Exercise 2.3. Fix a prime p and consider the metric space $(\mathbf{Q}, d_p)$ where d_p is the p-adic metric from Example 2.1.

(a) Let $p = 3$ and write down 4 elements of $\mathbf{B}_1(2)$ and 4 elements of $\mathbf{B}_{1/9}(3)$.

(b) Back to general prime p now: show that every triangle is isosceles. In other words, given three points in $\mathbf{Q}$, at least two of the three resulting (p-adic) distances are equal.

(c) Show that every point of an open ball is a centre. In other words, take an open ball $\mathbf{B}_r(c)$ with $r \in \mathbf{R}_{\geqslant 0}$ and $c \in \mathbf{Q}$ and suppose $x \in \mathbf{B}_r(c)$; prove that $\mathbf{B}_r(c) = \mathbf{B}_r(x)$.

(d) Show that given any two open balls, either one of them is contained in the other, or they are completely disjoint.

Solution. (a) We have

$$\left\{2, 5, -7, \frac{4}{5}\right\} \subseteq \mathbf{B}_1(2)$$
$$\left\{3, 30, -24, \frac{39}{4}\right\} \subseteq \mathbf{B}_{1/9}(3).$$

(b) Recall that in the proof of the triangle inequality for the p-adic metric in Example 2.1, the following stronger result was shown:

$$d_p(x, y) \leqslant \max\{d_p(x, t), d_p(t, y)\}.$$

with equality holding if $d_p(x, t) \neq d_p(t, y)$. But this precisely says that if $d_p(x, t) \neq d_p(t, y)$, then $d_p(x, y)$ has to be equal to the largest of $d_p(x, t)$ and $d_p(t, y)$.

(c) First $x \in \mathbf{B}_r(c)$ iff $c \in \mathbf{B}_r(x)$ (this is true for any metric space). So it suffices to show that $x \in \mathbf{B}_r(c)$ implies $\mathbf{B}_r(x) \subseteq \mathbf{B}_r(c)$. Let $y \in \mathbf{B}_r(x)$, then $d_p(y, x) < r$, so that

$$d_p(y, c) \leqslant \max\left\{d_p(y, x), d_p(x, c)\right\} < r,$$

in other words $y \in \mathbf{B}_r(c)$.

(d) Consider two open balls $\mathbf{B}_r(x)$ and $\mathbf{B}_t(y)$. Without loss of generality $r \leqslant t$. Suppose that the balls are not disjoint and let $z \in \mathbf{B}_r(x) \cap \mathbf{B}_t(y)$. By part (c) this implies that $\mathbf{B}_r(z) = \mathbf{B}_r(x)$ and $\mathbf{B}_t(z) = \mathbf{B}_t(y)$, so that

$$\mathbf{B}_r(x) = \mathbf{B}_r(z) \subseteq \mathbf{B}_t(z) = \mathbf{B}_t(y). \qquad \square$$

Exercise 2.4. Let $n \in \mathbf{N}$, $X = \mathbf{R}^n$ with the dot product $\cdot$, $\|x\| = \sqrt{x \cdot x}$ for $x \in X$, and $d(x, y) = \|x - y\|$ for $x, y \in X$. Then (X, d) is a metric space. (The function d is called the *Euclidean metric* or ℓ^2 *metric* on $\mathbf{R}^n$.)

[*Hint*: The Cauchy–Schwarz inequality can be useful for checking the triangle inequality.]

Solution. We have

(a) $d(x, y) = \|x - y\| = \sqrt{(x - y) \cdot (x - y)} = \sqrt{(-1)^2 (y - x) \cdot (y - x)} = \|y - x\| = d(y, x)$;

(b) Let $u = x - t$ and $v = t - y$, then we are looking to show that $\|u + v\| \leqslant \|u\| + \|v\|$. But:

$$\|u + v\|^2 = (u + v) \cdot (u + v) = \|u\|^2 + 2\, u \cdot v + \|v\|^2 \leqslant \|u\|^2 + 2\, |u \cdot v| + \|v\|^2$$
$$\leqslant \|u\|^2 + 2\, \|u\|\, \|v\| + \|v\|^2 = \left(\|u\| + \|v\|\right)^2,$$

where the last inequality sign comes from the Cauchy–Schwarz inequality.

(c) $d(x, y) = 0$ iff $(x - y) \cdot (x - y) = 0$ iff $x - y = 0$ iff $x = y$. $\square$

Exercise 2.5. Draw the unit open balls in the metric spaces $(\mathbf{R}^2, d_1)$ (Example 2.4), $(\mathbf{R}^2, d_2)$ (Exercise 2.4), and $(\mathbf{R}^2, d_\infty)$ (Example 2.5).

Solution. The Manhattan unit open ball is the interior of the square with vertices $(1,0)$, $(0,-1)$, $(-1,0)$, and $(0,1)$.

The Euclidean unit open ball is the interior of the unit circle centred at $(0,0)$.

The sup metric unit open ball is the interior of the square with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$. □

Exercise 2.6. Let X be a nonempty set and define

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that (X,d) is a metric space. (The function d is called the *discrete metric* on X.)

Solution. It is clear from the definition that $d(y,x) = d(x,y)$ and that $d(x,y) = 0$ iff $x = y$.

For the triangle inequality, take $x, y, t \in X$ and consider the different cases:

$x = y$	$x = t$	$t = y$	$d(x,y)$	$d(x,t) + d(t,y)$
True	True	True	0	$0 + 0 = 0$
True	False	False	0	$1 + 1 = 2$
False	True	False	1	$1 + 0 = 1$
False	False	True	1	$0 + 1 = 1$
False	False	False	1	$1 + 1 = 2$

In all cases we see that $d(x,y) \leqslant d(x,t) + d(t,y)$. □

Exercise 2.7. Let $n \in \mathbf{N}$, $X = \mathbf{F}_2^n$, and let $d(x,y)$ be the number of indices $i \in \{1,\dots,n\}$ such that $x_i \neq y_i$. Prove that (X,d) is a metric space. (The function d is called the *Hamming metric*.)

Solution. Do this from scratch if you want to, but I prefer to deduce it from other examples we have seen.

First look at the case $n = 1$, $X = \mathbf{F}_2$. Then $d(x,y)$ is precisely the discrete metric on $\mathbf{F}_2$ (see Exercise 2.6), in particular it is a metric. I'll denote it $d_{\mathbf{F}_2}$ for a moment to minimise confusion.

Back in the arbitrary $n \in \mathbf{N}$ case, note that $d(x,y)$ defined above can be expressed as

$$d(x,y) = d_{\mathbf{F}_2}(x_1,y_1) + \dots + d_{\mathbf{F}_2}(x_n,y_n),$$

which is a special case of Example 2.4, therefore also a metric. □

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