Tutorial Week 12

Topics: Convergence in the norm, maps of finite rank, spectrum

1. For $n \in \mathbb{N}$, consider the function $f_n \colon [0,2] \longrightarrow \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leqslant x \leqslant \frac{1}{n} \\ -n^2\left(x - \frac{2}{n}\right) & \text{if } \frac{1}{n} < x \leqslant \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} < x \leqslant 2. \end{cases}$$

(You might want to graph f_1, f_2, f_3 to get a feel for what the functions look like.)

Find the pointwise limit $f(x)$ of $\big(f_n(x)\big)$ for all $x \in [0,2]$.

Show that (f_n) does not converge to f with respect to the L^1 norm.

Solution. The pointwise limit is the constant function zero.

We have $\|f_n\|_{L^1} = 1$ for all $n \in \mathbb{N}$, so (f_n) does not converge to f with respect to the L^1 norm. □

2. Given a subset $S \subseteq [0,1]$, let $\mathbf{1}_S \colon [0,1] \longrightarrow \mathbb{R}$ denote the *characteristic function* of S, that is

$$\mathbf{1}_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

Consider the sequence of functions (g_n) defined as follows: write $n \in \mathbb{N}$ in the form

$$n = 2^k + \ell, \qquad k, \ell \in \mathbb{Z}_{\geqslant 0}, 0 \leqslant \ell < 2^k,$$

then define $g_n \colon [0,1] \longrightarrow \mathbb{R}$ by

$$g_n = \mathbf{1}_{[\ell/2^k, (\ell+1)/2^k]}.$$

(You might want to graph $g_1, \ldots, g_5$ to get a feel for what the functions look like.)

Show that (g_n) converges to the constant function zero with respect to the L^1 norm, but that $\big(g_n(x)\big)$ does not converge for any $x \in [0,1]$.

Solution. Note that $2^k \leqslant n < 2^{k+1}$, so that we have

$$\|g_n\|_{L^1} = \frac{1}{2^k} < \frac{2}{n} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty,$$

so $(g_n) \longrightarrow 0$ with respect to the L^1 norm.

However, for any $x \in [0,1]$ there are infinitely many values of n for which $g_n(x) = 0$ and infinitely many values of n for which $g_n(x) = 1$, which means that $\big(g_n(x)\big)$ does not converge. □

3. Let $R(H)$ denote the set of all maps $f \in B(H)$ of finite rank on a complex Hilbert space H.

Prove that $R(H)$ is a vector subspace of $B(H)$.

Solution. The constant zero map is certainly of finite rank.

If $f, g \in R(H)$ then $\operatorname{im}(f)$ and $\operatorname{im}(g)$ are finite-dimensional subspaces of H. Therefore $\operatorname{im}(f) + \operatorname{im}(g)$ is a finite-dimensional subspace of H, and certainly $\operatorname{im}(f+g) \subseteq \operatorname{im}(f) + \operatorname{im}(g)$.

If $f \in R(H)$ and $\alpha \in \mathbb{C}$ then $\operatorname{im}(\alpha f) \subseteq \operatorname{im}(f)$ is finite-dimensional. □

4. Prove that if $f \in R(H)$ and $g_1, g_2 \in B(H)$ then $g_2 \circ f \circ g_1 \in R(H)$.

   Solution. Clearly $\operatorname{im}(f \circ g_1) \subseteq \operatorname{im}(f)$ is finite-dimensional.

   On the other hand, $g_2 \circ f$ has a finite-dimensional domain, hence a finite-dimensional image. □

5. Prove that if $f \in R(H)$ then $f^* \in R(H)$.
   [*Hint*: Use Proposition 4.35.]

   Solution. By Proposition 4.35 we have, for all $x, y \in H$:

$$\begin{aligned}
\langle f(x), y \rangle &= \left\langle \sum_{i,j=1}^{m} c_{ij} \langle x, u_j \rangle u_i, y \right\rangle \\
&= \sum_{i,j=1}^{m} c_{ij} \langle x, u_j \rangle \langle u_i, y \rangle \\
&= \sum_{i,j=1}^{m} \langle x, \bar{c}_{ij} \langle y, u_i \rangle u_j \rangle \\
&= \left\langle x, \sum_{i,j=1}^{m} \bar{c}_{ij} \langle y, u_i \rangle u_j \right\rangle,
\end{aligned}$$

   from which we conclude that

$$f^*(y) = \sum_{i,j=1}^{m} \bar{c}_{ij} \langle y, u_i \rangle u_j \qquad \text{for all } y \in H,$$

   so f^* has finite rank. □

6. Recall the right shift operator $R \colon \ell^2 \longrightarrow \ell^2$

$$R(a_1, a_2, \dots) = (0, a_1, a_2, \dots).$$

   (a) Prove that R has no complex eigenvalues.

   (b) Prove that $0 \in \sigma(R)$.

   (c) Is R a compact map?

   Solution. (a) Suppose

$$(0, a_1, a_2, \dots) = R(a_1, a_2, \dots) = \lambda(a_1, a_2, \dots),$$

   then $\lambda a_1 = 0$, so either $\lambda = 0$ implying that $a_1 = a_2 = \cdots = 0$; or $a_1 = 0$ which implies that $a_2 = 0$, and so on. In both cases the alleged eigenvector is actually the zero vector.

   (b) It is clear that R is not surjective, hence not invertible, so $0 \in \sigma(R)$.

   (c) No, for the same reason that id_{ℓ^2} is not compact: $R(\mathbb{D}_1(0))$ contains $\{e_2, e_3, e_4, \dots\}$, hence a sequence that has no convergent subsequences. □

7. Let H be a complex Hilbert space and let

$$\mathrm{GL}(H) = \{f \in B(H) : f \text{ is invertible}\}.$$

For $f \in \mathrm{GL}(H)$, prove that

$$\mathbb{B}_r(f) \subseteq \mathrm{GL}(H) \qquad \text{where } r = \left\|f^{-1}\right\|^{-1}.$$

[*Hint*: Given $g \in \mathbb{B}_r(f)$, consider $i := -f^{-1} \circ (g - f)$ and use Proposition 4.40 to show that $\mathrm{id}_H - i$ is invertible.]

Conclude that $\mathrm{GL}(H)$ is an open subset of $B(H)$.

Solution. Take $g \in \mathbb{B}_r(f)$, then $\|g - f\| < r$. Let $i = -f^{-1} \circ (g - f)$, then

$$\|i\| = \|f^{-1} \circ (g - f)\| \leqslant \|f^{-1}\| \, \|g - f\| < \|f^{-1}\| r = 1,$$

so by Proposition 4.40 we get that $\mathrm{id}_H - i$ is invertible. But then

$$f \circ \left(\mathrm{id}_H - i\right) = f \circ \left(\mathrm{id}_H + f^{-1} \circ (g - f)\right) = f + g - f = g,$$

so g is the composition of two invertible maps, hence is itself invertible. □

8. Prove that the spectrum of any $f \in B(H)$ is a compact set.

[*Hint*: Use Question 7 to show that the resolvent $\rho(f)$ is an open subset of $\mathbb{C}$, then use Corollary 4.41.]

Solution. Consider the map $F_f : \mathbb{C} \longrightarrow B(H)$ given by

$$F_f(\lambda) = f - \lambda \, \mathrm{id}_H .$$

This is a continuous function (check this!), and $\rho(f) = F_f^{-1}\left(\mathrm{GL}(H)\right)$ is an open subset of $\mathbb{C}$, hence $\sigma(f)$ is a closed subset of $\mathbb{C}$. But by Corollary 4.41 $\sigma(f)$ is a subset of the compact disc (sic) $\mathbb{D}_{\|f\|}(0)$, so it is compact. □

9. Let V, W be normed spaces, with V Banach, and let $f \in B(V, W)$. Suppose that there exists a constant $c > 0$ such that

$$\|f(v)\|_W \geqslant c \, \|v\|_V \qquad \text{for all } v \in V.$$

Then $\mathrm{im}(f)$ is a closed subspace of W.

Solution. Let $w \in W$ and let (v_n) be a sequence in V such that $\left(f(v_n)\right) \longrightarrow w$ in W. We need to prove that $w \in \mathrm{im}(f)$.

For all $n, m \in \mathbb{N}$ we have

$$\|f(v_n) - f(v_m)\|_W = \|f(v_n - v_m)\|_W \geqslant c \, \|v_n - v_m\|_V .$$

But the sequence $\left(f(v_n)\right)$ converges, hence is Cauchy in W. Therefore the above inequality says that the sequence (v_n) is Cauchy in V. As V is Banach, we have $(v_n) \longrightarrow v \in V$. Since f is continuous, we have $w = \lim f(v_n) = f(v)$ and $w \in \mathrm{im}(f)$. □

10. Let $f \in B(H)$ be a self-adjoint map on a complex Hilbert space H and let $a + ib \in \mathbb{C}$. Prove that
$$\left\| \big(f - (a+ib)\,\mathrm{id}_H\big)(x) \right\| \geqslant |b|\,\|x\| \qquad \text{for all } x \in H.$$
[*Hint*: Expand $\|(f - (a+ib)\,\mathrm{id}_H)(x)\|^2$ using the inner product, take advantage of $f^* = f$, and manipulate until you get a sum of two squares, one of which is $b^2\|x\|^2$.]

Solution. We follow the hint:
$$\begin{aligned}
\left\| \big(f - (a+ib)\,\mathrm{id}_H\big) \right\|^2 &= \Big\langle \big(f - (a+ib)\,\mathrm{id}_H\big)(x), \big(f - (a+ib)\,\mathrm{id}_H\big)(x) \Big\rangle \\
&= \Big\langle \big(f - (a+ib)\,\mathrm{id}_H\big)(x), \big(f - (a-ib)\,\mathrm{id}_H\big)^*(x) \Big\rangle \\
&= \Big\langle \big(f - (a-ib)\,\mathrm{id}_H\big)\big(f - (a+ib)\,\mathrm{id}_H\big)(x), x \Big\rangle \\
&= \Big\langle \big((f - a\,\mathrm{id}_H)^2 + b^2\,\mathrm{id}_H\big)(x), x \Big\rangle \\
&= \Big\langle \big(f - a\,\mathrm{id}_H\big)^2(x), x \Big\rangle + b^2\|x\|^2 \\
&= \left\| (f - a\,\mathrm{id}_H)^2(x) \right\|^2 + b^2\|x\|^2 \\
&\geqslant b^2\|x\|^2.
\end{aligned}$$
□