Tutorial Week 11

Topics: Self-adjoint maps, uniform norm, pointwise and uniform convergence.

1. Let $f \in B(H,H)$ with H a Hilbert space. Then the maps

$$p = f^* \circ f \qquad \text{and} \qquad s = f + f^*$$

are self-adjoint.

Solution. Since f is continuous, the adjoint f^* is continuous, so the composition $p = f^* \circ f$ and the sum $s = f + f^*$ are both continuous.

Then

$$p^* = \left(f^* \circ f\right)^* = f^* \circ \left(f^*\right)^* = f^* \circ f = p$$
$$s^* = \left(f + f^*\right)^* = f^* + \left(f^*\right)^* = f^* + f = f + f^* = s.$$

□

2. The composition of two self-adjoint maps f, g on a Hilbert space is self-adjoint if and only if the maps commute.

Solution. We have

$$\langle f(g(x)), y\rangle = \langle g(x), f(y)\rangle = \langle x, g(f(y))\rangle$$

by the self-adjointness of f and g.

So $f \circ g$ is self-adjoint if and only if $g \circ f = f \circ g$, as claimed. □

3. Let $f \in B(H,H)$ with H a Hilbert space. Suppose that f is invertible with continuous inverse. Then the adjoint f^* is invertible and

$$\left(f^*\right)^{-1} = \left(f^{-1}\right)^*.$$

Solution. We want to prove that

$$\left(f^{-1}\right)^* \circ f^* = \mathrm{id}_H = f^* \circ \left(f^{-1}\right)^*.$$

We have for all $x, y \in H$:

$$\left\langle x, \left(f^{-1}\right)^*\left(f^*(y)\right)\right\rangle = \left\langle f^{-1}(x), f^*(y)\right\rangle = \left\langle f\left(f^{-1}(x)\right), y\right\rangle = \langle x, y\rangle$$

implying that $\left(f^{-1}\right)^* \circ f^* = \mathrm{id}_H$, and similarly for the other composition. □

4. Let B be an orthonormal system in a Hilbert space H. Prove that B is an orthonormal basis if and only if:

$$\text{for every } x \in H, \text{ if } \langle x, y\rangle = 0 \text{ for all } y \in B, \text{ then } x = 0.$$

Solution. By definition, B is an orthonormal basis if and only if $\overline{\mathrm{Span}(B)} = H$. So given $x \in H$ we have

$$\begin{aligned}\langle x, y\rangle = 0 \qquad \text{for all } y \in B \quad &\iff \quad x \in B^\perp \\ &\iff \quad x \in \overline{\mathrm{Span}(B)}^\perp\end{aligned}$$

and

$$x = 0 \quad \iff \quad x \in H^\perp,$$

hence the required statement. □

5. For each $n \in \mathbb{N}$ define $f_n \colon [0,1] \longrightarrow \mathbb{R}$ by

$$f_n(x) = \frac{x^2}{1+nx}.$$

Convince yourself that each f_n is continuous.

Find the pointwise limit f of the sequence (f_n) and determine whether the sequence converges uniformly to f.

Solution. We know that x^2 is continuous on $[0,1]$ and $1+nx$ is continuous and nonzero on $[0,1]$, so their quotient f_n is continuous on $[0,1]$.

At $x=0$ we have $f_n(0)=0$ so $f(0)=0$.

If $x \in (0,1]$ then

$$|f_n(x)| = \frac{x^2}{1+nx} \leqslant \frac{1}{1+nx} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

so $f(x) = 0$ for all $x \in (0,1]$.

So the pointwise limit is the constant function 0 on $[0,1]$.

To check whether the convergence is uniform we look at

$$\|f_n - f\| = \|f_n\| = \sup_{x\in[0,1]} |f_n(x)| = \sup_{x\in[0,1]} \frac{x^2}{1+nx}.$$

As $[0,1]$ is compact, f_n attains its supremum as a global maximum on $[0,1]$. Since f_n is differentiable on $(0,1)$ we can use its derivative to look for local maxima:

$$f_n'(x) = \frac{x(nx+2)}{(1+nx)^2},$$

and since $nx+2>0$ on $[0,1]$, the maximum must occur at one of the boundary points:

$$f_n(0) = 0 \qquad \text{and} \qquad f_n(1) = \frac{1}{1+n},$$

so

$$\|f_n\| = \frac{1}{1+n}.$$

This converges to 0 as $n \longrightarrow \infty$, so $(f_n) \longrightarrow f$ uniformly. □

6. For each $n \in \mathbb{N}$ define $f_n \colon [0,1] \longrightarrow \mathbb{R}$ by

$$f_n(x) = \frac{1-x^n}{1+x^n}.$$

Convince yourself that each f_n is continuous.

Find the pointwise limit f of the sequence (f_n) and determine whether the sequence converges uniformly to f.

Solution. Both $1-x^n$ and $1+x^n$ are continuous, and $1+x^n$ is nonzero on $[0,1]$, so their quotient f_n is continuous on $[0,1]$.

Note that at $x=1$ we have $f_n(1) = \frac{0}{2} = 0$, so $f(1)=0$.

But if $x < 1$ then $(x^n) \longrightarrow 0$ as $n \longrightarrow \infty$, so that

$$f_n(x) = \frac{1 - x^n}{1 + x^n} \longrightarrow \frac{1}{1} = 1,$$

and so $f(x) = 1$. In summary:

$$f(x) = \begin{cases} 1 & \text{if } 0 \leqslant x < 1 \\ 0 & \text{if } x = 1. \end{cases}$$

As the f_n are all continuous, so if $(f_n) \longrightarrow f$ uniformly then f would be continuous. Since that is not the case, the convergence is not uniform. □

7. Suppose that the Weierstraß Approximation Theorem holds for $K = [0, 1]$.

   Prove that the Theorem holds for any closed interval $[a, b]$ with $a < b$.

   [*Hint*: Find a polynomial function of degree one $\varphi\colon [0, 1] \longrightarrow [a, b]$ that is surjective and use it and its inverse to move between functions on $[0, 1]$ and functions on $[a, b]$.]

   Solution. Consider $\varphi\colon [0, 1] \longrightarrow [a, b]$ given by

$$\varphi(x) = (1 - x)a + xb.$$

   It is clearly continuous and has inverse $\psi\colon [a, b] \longrightarrow [0, 1]$ given by

$$\psi(y) = \frac{y - a}{b - a},$$

   also clearly continuous.

   Now if $f \in \mathrm{Cts}([a, b], \mathbb{R})$, then $f \circ \varphi \in \mathrm{Cts}([0, 1], \mathbb{R})$, so there is a sequence of polynomials (p_n) with $p_n\colon [0, 1] \longrightarrow \mathbb{R}$ that converges to $f \circ \varphi$ in the uniform norm.

   Let $q_n = p_n \circ \psi$; as the composition of two polynomials, it is a polynomial $q_n\colon [a, b] \longrightarrow \mathbb{R}$. We have

$$\begin{aligned} \|q_n - f\| &= \sup_{y \in [a,b]} |q_n(y) - f(y)| \\ &= \sup_{y \in [a,b]} |p_n(\psi(y)) - f(y)| \\ &= \sup_{x \in [0,1]} |p_n(\psi(\varphi(x))) - f(\varphi(x))| \\ &= \sup_{x \in [0,1]} |p_n(x) - f(\varphi(x))| \\ &= \|p_n - f \circ \varphi\| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \end{aligned}$$

   □

8. (*) Prove that for any $x \in \mathbb{R}$ and for any $n \in \mathbb{Z}_{\geqslant 0}$ we have

   (a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} = 1;$

   (b) $\displaystyle\sum_{k=0}^{n} k \binom{n}{k} x^k (1 - x)^{n-k} = nx;$

   (c) $\displaystyle\sum_{k=0}^{n} k^2 \binom{n}{k} x^k (1 - x)^{n-k} = n(n - 1)x^2 + nx;$

   (d) $\displaystyle\delta^2 \sum_{k\colon |k/n - x| \geqslant \delta} \binom{n}{k} x^k (1 - x)^{n-k} \leqslant \frac{1}{4n}$ for all $\delta > 0$.

[*Hint*: For (b), note that $k\binom{n}{k} = n\binom{n-1}{k-1}$.
For (c), start by showing that $\sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} = n(n-1)x^2$.
For (d), use the fact that $\delta^2 \leqslant \big(x - k/n\big)^2$ for all k such that $|k/n - x| \geqslant \delta$, so that the sum in question is bounded above by $\sum_{k=0}^{n} \big(x - k/n\big)^2 \binom{n}{k} x^k (1-x)^{n-k}$.]

Solution. (a) This follows from the binomial theorem:

$$1 = \big(x + (1-x)\big)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}.$$

(b) As hinted, we have

$$k\binom{n}{k} = \frac{kn \cdot (n-1)!}{k!\,(n-k)!} = n\,\frac{(n-1)!}{(k-1)!\big((n-1)-(k-1)\big)!} = n\binom{n-1}{k-1}.$$

Therefore

$$\sum_{k=0}^{n} k\binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} x^k (1-x)^{n-k} = nx \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j} = nx,$$

where we used the substitution $j = k-1$, and at the end the result of the previous part.

(c) Iterating the previous part a second time, we have

$$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2},$$

after which we evaluate

$$\sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} = n(n-1)x^2 \sum_{j=0}^{n-2} \binom{n-2}{j} x^j (1-x)^{n-2-j} = n(n-1)x^2,$$

and finally conclude by combining this with the result of the previous part.

(d) Starting with the hint:

$$\begin{aligned}
\delta^2 \sum_{k\,:\,|k/n-x|\geqslant\delta} \binom{n}{k} x^k (1-x)^{n-k} &\leqslant \sum_{k=0}^{n} \big(x - k/n\big)^2 \binom{n}{k} x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n} \left(x^2 - \frac{2x}{n}k + \frac{1}{n^2}k^2\right) \binom{n}{k} x^k (1-x)^{n-k} \\
&= x^2 - \frac{2x}{n}nx + \frac{1}{n^2}\big(n(n-1)x^2 + nx\big) \\
&= \frac{x(1-x)}{n} \\
&\leqslant \frac{1}{4n}.
\end{aligned}$$

Along the way we used the results of parts (a), (b), and (c), together with the easy fact that the global maximum of $x(1-x)$ for $x \in [0,1]$ is $1/4$. □