Tutorial Week 10

Topics: Projections; adjoint maps

1. Let V be a normed space and φ, ψ be commuting projections: $\varphi \circ \psi = \psi \circ \varphi$. Prove that $\varphi \circ \psi$ is a projection with image $\operatorname{im} \varphi \cap \operatorname{im} \psi$.

   Solution. We know that the composition of continuous linear maps is continuous linear, so this is true for $\varphi \circ \psi$. To conclude that it is a projection, we need to compute its square:

   $$(\varphi \circ \psi) \circ (\varphi \circ \psi) = (\varphi \circ \varphi) \circ (\psi \circ \psi) = \varphi \circ \psi,$$

   where it was crucial that φ and ψ commute.

   For the statement about the image, note that $w \in \operatorname{im}(\varphi \circ \psi)$ if and only if there exists $v \in V$ such that

   $$w = \varphi(\psi(v)) = \psi(\varphi(v)),$$

   which implies that $w \in \operatorname{im} \varphi \cap \operatorname{im} \psi$. So $\operatorname{im}(\varphi \circ \psi) \subseteq \operatorname{im} \varphi \cap \operatorname{im} \psi$.

   Conversely, suppose $w \in \operatorname{im} \varphi \cap \operatorname{im} \psi$, then there exists $v \in V$ such that $w = \psi(v)$. But $w \in \operatorname{im} \varphi$ and φ is a projection, so that

   $$w = \varphi(w) = \varphi(\psi(v)) \in \operatorname{im}(\varphi \circ \psi).$$ □

2. Let φ be a nonzero orthogonal projection (that is, φ is not the constant function 0) on an inner product space V. Prove that $\|\varphi\| = 1$.

   Solution. We know that $(\operatorname{im} \varphi)^{\perp} = \ker \varphi$. For any x we have

   $$\varphi(x - \varphi(x)) = \varphi(x) - \varphi^2(x) = \varphi(x) - \varphi(x) = 0,$$

   so $x - \varphi(x) \in \ker \varphi$. Therefore

   $$\langle x, \varphi(x) \rangle - \|\varphi(x)\|^2 = \langle x - \varphi(x), \varphi(x) \rangle = 0,$$

   so

   $$\|\varphi(x)\|^2 = \langle x, \varphi(x) \rangle \leqslant \|x\| \, \|\varphi(x)\|$$

   by the Cauchy–Schwarz Inequality. Hence $\|\varphi(x)\| \leqslant \|x\|$ for all x, hence $\|\varphi\| \leqslant 1$. However for $x \in \operatorname{im} \varphi$ we have $\varphi(x) = x$ so $\|\varphi(x)\| = \|x\|$ and we conclude that $\|\varphi\| = 1$. □

3. Let S be a subset of a Hilbert space H. Prove that $\operatorname{Span}(S)$ is dense in H if and only if $S^{\perp} = 0$.

   Solution. If $S^{\perp} = 0$ then

   $$\overline{\operatorname{Span}(S)} = (S^{\perp})^{\perp} = 0^{\perp} = H.$$

   Conversely, if S is dense in H then

   $$S^{\perp} = \overline{\operatorname{Span}(S)}^{\perp} = H^{\perp} = 0.$$ □

4. Let V, W be inner product spaces and let $f \in B(V, W)$. Prove that

   $$\|f\| = \sup_{\|v\|_V = \|w\|_W = 1} |\langle f(v), w \rangle_W|.$$

   [*Hint*: Use Exercise 4.2 which says that $\|v\| = \sup_{\|w\|=1} |\langle v, w \rangle|$.]

Solution. Recall from Exercise 4.2 that

$$\|u\|_W = \sup_{\|w\|_W=1} \left|\langle u, w\rangle_W\right| \qquad \text{for all } u \in W.$$

Setting $u = f(v)$ for some $v \in V$, we get

$$\|f(v)\|_W = \sup_{\|w\|_W=1} \left|\langle f(v), w\rangle_W\right| \qquad \text{for all } v \in V.$$

Therefore

$$\|f\| = \sup_{\|v\|_V=1} \|f(v)\|_W = \sup_{\|v\|_V=\|w\|_W=1} \left|\langle f(v), w\rangle_W\right|.$$

□

5. Recall that the adjoint $f^*\colon Y \longrightarrow X$ of a continuous linear map $f\colon X \longrightarrow Y$ of Hilbert spaces satisfies the property

$$\big\langle f(x), y\big\rangle_Y = \big\langle x, f^*(y)\big\rangle_X \qquad \text{for all } x \in X, y \in Y.$$

Prove that for all $\alpha \in \mathbb{F}$ we have

$$\big(\alpha f\big)^* = \overline{\alpha} f^*.$$

Solution. We have

$$\begin{aligned}
\big\langle x, (\alpha f)^*(y)\big\rangle &= \big\langle (\alpha f)(x), y\big\rangle \\
&= \alpha\big\langle f(x), y\big\rangle \\
&= \alpha\big\langle x, f^*(y)\big\rangle \\
&= \big\langle x, \overline{\alpha}\, f^*(y)\big\rangle.
\end{aligned}$$

□

6. Given continuous linear maps $g\colon X \longrightarrow Y$ and $f\colon Y \longrightarrow Z$ of Hilbert spaces, prove that

$$\big(f \circ g\big)^* = g^* \circ f^*.$$

Solution. We have

$$\begin{aligned}
\big\langle x, (f \circ g)^*(y)\big\rangle &= \big\langle (f \circ g)(x), y\big\rangle \\
&= \big\langle f\big(g(x)\big), y\big\rangle \\
&= \big\langle g(x), f^*(y)\big\rangle \\
&= \big\langle x, g^*\big(f^*(y)\big)\big\rangle \\
&= \big\langle x, \big(g^* \circ f^*\big)(y)\big\rangle.
\end{aligned}$$

□

7. Prove that for any Hilbert space X we have

$$\mathrm{id}_X^* = \mathrm{id}_X\,.$$

Solution. Tautological:

$$\big\langle \mathrm{id}_X(x), y\big\rangle = \big\langle x, y\big\rangle = \big\langle x, \mathrm{id}_X(y)\big\rangle.$$

□

8. Prove that for any continuous linear map $f\colon X \longrightarrow Y$ of Hilbert spaces, we have

$$\big(f^*\big)^* = f.$$

Solution. We have

$$\begin{aligned} \langle x, (f^*)^*(y)\rangle &= \langle f^*(x), y\rangle \\ &= \overline{\langle y, f^*(x)\rangle} \\ &= \overline{\langle f(y), x\rangle} \\ &= \langle x, f(y)\rangle. \end{aligned}$$

□

9. Let $f\colon X \longrightarrow Y$ be a continuous linear map of Hilbert spaces. Prove that

$$\ker(f^*) = (\operatorname{im} f)^{\perp} \qquad \text{and} \qquad \overline{\operatorname{im}(f^*)} = (\ker f)^{\perp}.$$

Solution. We have

$$\begin{aligned} y \in (\operatorname{im} f)^{\perp} &\iff y \perp f(x) \qquad \text{for all } x \in X \\ &\iff \langle f(x), y\rangle = 0 \qquad \text{for all } x \in X \\ &\iff \langle x, f^*(y)\rangle = 0 \qquad \text{for all } x \in X \\ &\iff f^*(y) = 0 \\ &\iff y \in \ker f^*. \end{aligned}$$

From this and Exercise 4.12 we have

$$\ker f = \ker(f^*)^* = (\operatorname{im} f^*)^{\perp},$$

so that

$$(\ker f)^{\perp} = ((\operatorname{im} f^*)^{\perp})^{\perp} = \overline{\operatorname{im} f^*},$$

where the last equality comes from Corollary 4.13. □

10. Consider the function $g\colon \ell^2 \longrightarrow \mathbb{F}$ given by

$$g(x) = \sum_{n=1}^{\infty} \frac{x_n}{n^2}.$$

(a) Find $y \in \ell^2$ such that

$$g(x) = \langle x, y\rangle \qquad \text{for all } x \in \ell^2.$$

(b) Deduce that g is linear and bounded and find its norm $\|g\|$.

[*Hint*: You may use without proof the fact that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.]

Solution. (a) Setting $y = (y_n)$ with

$$y_n = \frac{1}{n^2},$$

we certainly have for all $x = (x_n) \in \ell^2$:

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y}_n = \sum_{n=1}^{\infty} \frac{x_n}{n^2} = g(x).$$

We should check that $y \in \ell^2$:

$$\|y\|_{\ell^2}^2 = \sum_{n=1}^{\infty} y_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(b) From the previous part we know that $g = y^\vee$, so certainly g is linear and bounded. We also have

$$\|g\| = \|y^\vee\| = \|y\|_{\ell^2} = \frac{\pi^2}{3\sqrt{10}},$$

as we have seen in the previous part.

□