Tutorial Week 09

Topics: more sequence spaces; inner product spaces.

1. Consider the map $\pi_1 \colon \mathbb{F}^{\mathbb{N}} \longrightarrow \mathbb{F}$ given by

$$\pi_1\big((a_n)\big) = a_1.$$

(a) Show that π_1 is linear.

(b) Prove that the restriction of π_1 to ℓ^∞ or to ℓ^p for $p \geqslant 1$ is continuous and surjective.

Solution.

(a) Straightforward.

(b) We have for $a \in \ell^\infty$:

$$|\pi_1(a)| = |a_1| \leqslant \sup_{n \geqslant 1}\{|a_n|\} = \|a\|_{\ell^\infty},$$

so π_1 is bounded.

Similarly for $a \in \ell^p$:

$$|\pi_1(a)| = |a_1| = \big(|a_1|^p\big)^{1/p} \leqslant \left(\sum_{n \geqslant 1} |a_n|^p\right)^{1/p} = \|a\|_{\ell^p}.$$

For the surjectivity we note that for any $a \in \mathbb{F}$ we have $\pi_1\big((a, 0, 0, \dots)\big) = a$ and $(a, 0, 0 \dots) \in \ell^1 \subseteq \ell^p$ for all $p \geqslant 1$ and for $p = \infty$. □

2. Consider the left shift map $L \colon \mathbb{F}^{\mathbb{N}} \longrightarrow \mathbb{F}^{\mathbb{N}}$ given by $L\big((a_n)\big) = (a_{n+1})$, that is

$$L(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots).$$

(a) Prove that L is a surjective linear map. What is the kernel of L?

(b) Prove that for all $p \geqslant 1$ and for $p = \infty$, the restriction of L to ℓ^p is a surjective continuous map onto ℓ^p.

(c) Define the right shift map $R \colon \mathbb{F}^{\mathbb{N}} \longrightarrow \mathbb{F}^{\mathbb{N}}$ and prove that it is an injective linear map, the restriction of which is distance-preserving for any ℓ^p with $p \geqslant 1$ and $p = \infty$.

(d) Check that $L \circ R = \mathrm{id}_{\mathbb{F}^{\mathbb{N}}} \neq R \circ L$.

Solution.

(a) It is clear that L is surjective. Linearity is pretty straightforward, and it's also clear that $\ker(L) = \mathrm{Span}\{e_1\}$.

(b) We have

$$\|L(a_1, a_2, a_3, \dots)\|_{\ell^p} = \left(\sum_{n=2}^{\infty} |a_n|^p\right)^{1/p} \leqslant \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} = \big\|(a_1, a_2, \dots)\big\|_{\ell^p},$$

so L is bounded, and $L\big((a_n)\big) \in \ell^p$ if $(a_n) \in \ell^p$.

For the surjectivity note that if $b = (b_1, b_2, \dots) \in \ell^p$, then

$$b = L(a) \qquad \text{for } a = (0, b_1, b_2, \dots)$$

and $\|a\|_{\ell^p} = \|b\|_{\ell^p}$, so $a \in \ell^p$.

The case of ℓ^∞ is done in a similar way.

(c) To get a linear map we need to set

$$R(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots).$$

Both injectivity and linearity are straightforward.

We have, for $p \geqslant 1$ or $p = \infty$:

$$\|R(a_1, a_2, \dots)\|_{\ell^p} = \|(0, a_1, a_2, \dots)\|_{\ell^p} = \|(a_1, a_2, \dots)\|_{\ell^p},$$

so R is distance-preserving and $R(a) \in \ell^p$ if $a \in \ell^p$.

(d) Clear. For any $a = (a_n) \in \mathbb{F}^{\mathbb{N}}$ we have

$$\begin{aligned} L(R(a)) &= L(R(a_1, a_2, \dots)) = L(0, a_1, a_2, \dots) = (a_1, a_2, \dots) = a, \\ R(L(a)) &= R(L(a_1, a_2, \dots)) = R(a_2, a_3, \dots) = (0, a_2, a_3, \dots) \neq a \text{ unless } a_1 = 0. \end{aligned}$$ □

3. Consider the subset c of $\mathbb{F}^{\mathbb{N}}$ consisting of all convergent sequences (with any limit).
   (a) Convince yourself that c is a vector subspace of ℓ^∞.
   (b) Prove that $\lim\colon c \longrightarrow \mathbb{F}$ given by

$$(a_n) \longmapsto \lim_{n \longrightarrow \infty} (a_n)$$

   is a continuous surjective linear map.

   (c) Prove that the formula

$$J\big((a_n)\big) = R\big((a_n)\big) - \Big(\lim_{n \longrightarrow \infty} a_n\Big)(1, 1, \dots)$$

   defines a linear homeomorphism $J\colon c \longrightarrow c_0$. (Here R denotes the right shift map.)

   (d) Show that c is separable and find a Schauder basis for c.

Solution. (a) We know that convergent sequences are bounded, so $c \subseteq \ell^\infty$. We also know that the sum of two convergent sequences is convergent, and that a scalar multiple of a convergent sequence is convergent, and that the constant sequence $(0, 0, \dots)$ is convergent.

(b) We know that lim is linear, as a consequence of the continuity of addition and of scalar multiplication.

It is certainly surjective, as given any $a \in \mathbb{F}$ the constant sequence $(a, a, \dots)$ converges to a.

Finally, if $a = (a_n) \in c$ then (a_n) is a bounded sequence and

$$\Big|\lim_{n \longrightarrow \infty} a_n\Big| \leqslant \sup_{n \in \mathbb{N}} |a_n| = \|a\|_{\ell^\infty},$$

so lim is a bounded linear map.

(c) It is clear that J is linear and continuous, as R and lim are linear and continuous.

We exhibit an explicit inverse of J: let $K\colon c_0 \longrightarrow c$ be given by

$$K\big((b_n)\big) = L\big((b_n)\big) - b_1(1, 1, \dots).$$

Note that K is linear and continuous, as L and $(b_n) \longmapsto b_1$ are linear and continuous.

We check that K and J and inverses. If $b \in c_0$ and $a \in c$ then:

$$\begin{aligned}
J\big(K(b)\big) &= J\big(L(b)\big) - b_1 J(1,1,\dots) \\
&= R\big(L(b)\big) - 0(1,1,\dots) - b_1\big(R(1,1,\dots) - (1,1,\dots)\big) \\
&= (0, b_2, b_3, \dots) - b_1(-1,0,0,\dots) \\
&= b, \\
K\big(J(a)\big) &= K\big(R(a)\big) - \big(\lim a_n\big)K(1,1,\dots) \\
&= L\big(R(a)\big) - \big(\lim a_n\big)\big(L(1,1,\dots) - (1,1,\dots)\big) \\
&= a.
\end{aligned}$$

(d) We know that $\{e_1, e_2, e_3, \dots\}$ is a Schauder basis for c_0, so we apply $K\colon c_0 \longrightarrow c$ to this to get:

$$\begin{aligned}
K(e_1) &= L(e_1) - (1,1,\dots) = -(1,1,\dots) \\
K(e_2) &= L(e_2) - 0(1,1,\dots) = e_1 \\
K(e_3) &= L(e_3) - 0(1,1,\dots) = e_2 \\
&\vdots \\
K(e_n) &= L(e_n) - 0(1,1,\dots) = e_{n-1} \qquad \text{for } n \geqslant 2 \\
&\vdots
\end{aligned}$$

We suspect then that $\{(1,1,\dots), e_1, e_2, e_3, \dots\}$ is a Schauder basis for c.

This is of course true whenever we have a linear homeomorphism $f\colon V \longrightarrow W$ between normed spaces: If $\{b_1, b_2, \dots\}$ is a Schauder basis for V, then $\{f(b_1), f(b_2), \dots\}$ is a Schauder basis for W.

Let $w \in W$ and let $v = f^{-1}(w) \in V$. Write

$$v = \sum_{j\in\mathbb{N}} \alpha_j b_j,$$

then

$$w = f(v) = \sum_{j\in\mathbb{N}} \alpha_j f(b_j).$$

Uniqueness follows from the uniqueness of the expansion for v. □

4. For any $n \in \mathbb{N}$, give a linear distance-preserving map $\mathbb{F}^n \longrightarrow \ell^2$. (Take the Euclidean norm on $\mathbb{F}^n$.)

Solution. Consider $f\colon \mathbb{F}^n \longrightarrow \ell^2$ given by

$$f(a) = f(a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, a_n, 0, 0, \dots).$$

We have

$$\big\|(a_1, a_2, \dots, a_n, 0, 0, \dots)\big\|_{\ell^2} = \left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2} = \|(a_1, a_2, \dots, a_n)\|_{\mathbb{F}^n},$$

so $f(a) \in \ell^2$, and f is distance-preserving.

Linearity is straightforward. □

5. Let $\big(V, \langle\cdot,\cdot\rangle\big)$ be an inner product space. Prove that the inner product is a continuous function.

Solution. One way is to use the Polarisation Identity and the fact that the norm is continuous.

But we can also proceed more directly: suppose $(x_n, y_n) \longrightarrow (x, y)$, then $(x_n) \longrightarrow x$ and $(y_n) \longrightarrow y$. As (y_n) converges, it is bounded, so there exists $C \geqslant 0$ such that $\|y_n\| \leqslant C$ for all $n \in \mathbb{N}$.

Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that

$$\|x_n - x\| < \frac{\varepsilon}{2C} \qquad \text{and} \qquad \|y_n - y\| < \frac{\varepsilon}{2\|x\|} \text{ for all } n \geqslant N.$$

Then

$$\begin{aligned}
\left|\langle x_n, y_n\rangle - \langle x, y\rangle\right| &= \left|\langle x_n, y_n\rangle - \langle x, y_n\rangle + \langle x, y_n\rangle - \langle x, y\rangle\right| \\
&= \left|\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle\right| \\
&\leqslant \left|\langle x_n - x, y_n\rangle\right| + \left|\langle x, y_n - y\rangle\right| \\
&\leqslant \|x_n - x\|\,\|y_n\| + \|x\|\,\|y_n - y\| \\
&\leqslant C\|x_n - x\| + \|x\|\,\|y_n - y\| \\
&< \varepsilon.
\end{aligned}$$

We conclude that $\big(\langle x_n, y_n\rangle\big) \longrightarrow \langle x, y\rangle$. □

6. Let $\big(V, \langle\cdot,\cdot\rangle\big)$ be an inner product space. For any $v \in V$ we have

$$\|v\| = \sup_{\|w\|=1} \left|\langle v, w\rangle\right|.$$

The supremum is in fact achieved by a well-chosen w.

Solution. If $v = 0$ then the equality is obvious.

So assume now that $v \neq 0$. By Cauchy–Schwarz we have for all $w \in V$:

$$\left|\langle v, w\rangle\right| \leqslant \|v\|\,\|w\|.$$

Therefore for all $w \in V$ with $\|w\| = 1$ we have

$$\left|\langle v, w\rangle\right| \leqslant \|v\|,$$

so that

$$\sup_{\|w\|=1} \left|\langle v, w\rangle\right| \leqslant \|v\|.$$

To get equality, take $w = \frac{1}{\|v\|}v$ and see that the LHS is indeed $\|v\|$. □

7. Let $\big(V, \langle\cdot,\cdot\rangle\big)$ be an inner product space and let R, S be subsets of V.
   (a) Prove that $S \cap S^\perp = 0$.
   (b) Prove that if $R \subseteq S$ then $S^\perp \subseteq R^\perp$.
   (c) Prove that $S \subseteq \big(S^\perp\big)^\perp$.
   (d) Prove that $S^\perp = \overline{\mathrm{Span}(S)}^\perp$.

Solution.

(a) If $x \in S^{\perp} \cap S$ then $\langle x, s\rangle = 0$ for all $s \in S$, in particular $\langle x, x\rangle = 0$ so $x = 0$.

(b) Suppose $R \subseteq S$ and $x \in S^{\perp}$. For any $r \in R$ we have $r \in S$ so $\langle x, r\rangle = 0$, hence $x \in R^{\perp}$.

(c) Let $s \in S$. For any $x \in S^{\perp}$, we have

$$\langle s, x\rangle = \overline{\langle x, s\rangle} = 0,$$

so $s \in \left(S^{\perp}\right)^{\perp}$.

(d) Since $S \subseteq \mathrm{Span}(S) \subseteq \overline{\mathrm{Span}(S)}$, by part (b) we get

$$\overline{\mathrm{Span}(S)}^{\perp} \subseteq S^{\perp}.$$

In the other direction, suppose $x \in S^{\perp}$. For any $v \in \mathrm{Span}(S)$ we have

$$\langle x, v\rangle = \langle x, \alpha_1 s_1 + \cdots + \alpha_n s_n\rangle = \overline{\alpha}_1\langle x, s_1\rangle + \cdots + \overline{\alpha}_n\langle x, s_n\rangle = 0.$$

Now if $(v_n) \longrightarrow w \in \overline{\mathrm{Span}(S)}$ with $v_n \in \mathrm{Span}(S)$, we have

$$\langle x, w\rangle = \langle x, \lim v_n\rangle = \lim\langle x, v_n\rangle = \lim 0 = 0.$$ □

8. Let (X, d) be a metric space and let $S \subseteq X$. Prove that $d_S(x) = 0$ if and only if $x \in \overline{S}$.

Solution. Suppose $0 = d_S(x) = \inf_{s \in S} d(s, x)$, then there exists a sequence (s_n) with $s_n \in S$ and $d(s_n, x) \longrightarrow d_S(x) = 0$, so $(s_n) \longrightarrow x$, so $x \in \overline{S}$.

Conversely, if $x \in \overline{S}$ then there exists a sequence $(s_n) \longrightarrow x$, so

$$d_S(x) = \inf_{s \in S} d_S(x) \leqslant \inf_{n \in \mathbb{N}} d(s_n, x) = 0.$$ □