Tutorial Week 08

Topics: series in normed spaces; sequence spaces.

1. If a series in a normed space $\big(V, \|\cdot\|\big)$

$$\sum_{n=1}^{\infty} a_n$$

converges, and converges absolutely, then

$$\left\|\sum_{n=1}^{\infty} a_n\right\| \leqslant \sum_{n=1}^{\infty} \|a_n\|.$$

Solution. This follows from the usual triangle inequality.

For any $m \in \mathbb{N}$, we have

$$\|a_1 + \cdots + a_m\| \leqslant \|a_1\| + \cdots + \|a_m\|.$$

Taking limits as $m \longrightarrow \infty$ we get

$$\left\|\sum_{n=1}^{\infty} a_n\right\| = \left\|\lim_{m\longrightarrow\infty} \sum_{n=1}^{m} a_n\right\| = \lim_{m\longrightarrow\infty} \left\|\sum_{n=1}^{m} a_n\right\| \leqslant \lim_{m\longrightarrow\infty} \sum_{n=1}^{m} \|a_n\| = \sum_{n=1}^{\infty} \|a_n\|.$$ □

2. Give an example of a series that converges but does not converge absolutely.

Solution. In $\mathbb{R}$, consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$ □

3. If $f \in B(V, W)$ with V, W normed spaces, and the series

$$\sum_{n=1}^{\infty} \alpha_n v_n, \qquad \alpha_n \in \mathbb{F}, v_n \in V,$$

converges in V, then the series

$$\sum_{n=1}^{\infty} \alpha_n f(v_n)$$

converges in W to the limit

$$f\left(\sum_{n=1}^{\infty} \alpha_n v_n\right).$$

Solution. Let

$$x_m = \sum_{n=1}^{m} \alpha_n v_n, \qquad x = \sum_{n=1}^{\infty} \alpha_n v_n.$$

We know that $(x_m) \longrightarrow x$ in V.

Since $f \in B(V, W)$ is continuous, we have that $\big(f(x_m)\big) \longrightarrow f(x)$ in W.

But f is also linear, so

$$f(x_m) = \sum_{n=1}^{m} \alpha_n f(v_n).$$

Hence

$$\left(\sum_{n=1}^{m} \alpha_n f(v_n)\right) \longrightarrow f(x),$$

so that the series

$$\sum_{n=1}^{\infty} \alpha_n f(v_n) \quad \text{converges to } f(x).$$ □

4. Prove that if $u = \big(u_n\big) \in \ell^\infty$ and $v = \big(v_n\big) \in \ell^1$, then

$$\sum_{n=1}^{\infty} |u_n v_n| \leqslant \|u\|_{\ell^\infty} \|v\|_{\ell^1}.$$

Solution. Very straightforward.

By the definition of ℓ^∞ and the ℓ^∞-norm, we have $|u_n| \leqslant \|u\|_{\ell^\infty}$ for all $n \in \mathbb{N}$. Therefore for any $m \in \mathbb{N}$ we have

$$\sum_{n=1}^{m} |u_n v_n| \leqslant \|u\|_\infty \sum_{n=1}^{m} |v_n|,$$

but the latter series converges because $v \in \ell^1$, to $\|v\|_{\ell^1}$ and we get

$$\sum_{n=1}^{\infty} |u_n v_n| \leqslant \|u\|_{\ell^\infty} \|v\|_{\ell^1}.$$

□

5. Consider the subset $c_0 \subseteq \mathbb{F}^{\mathbb{N}}$ of all sequences with limit 0:

$$c_0 = \big\{ (a_n) \in \mathbb{F}^{\mathbb{N}} \colon (a_n) \longrightarrow 0 \big\}.$$

Prove that c_0 is a closed subspace of ℓ^∞.

Conclude that c_0 is a Banach space.

Solution. It's pretty clear that c_0 is a subspace of $\mathbb{F}^{\mathbb{N}}$, and hence of ℓ^∞. To show that c_0 is closed in ℓ^∞, let $(x_n) \longrightarrow x \in \ell^\infty$ with $x_n \in c_0$ for all $n \in \mathbb{N}$. We want to prove that $x \in c_0$.

Write $x_n = \big(a_{nm}\big) = \big(a_{n1}, a_{n2}, a_{n3}, \dots\big)$ and $x = (a_m) = (a_1, a_2, a_3, \dots)$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have

$$\sup_m |a_m - a_{nm}| = \|x - x_n\|_{\ell^\infty} < \frac{\varepsilon}{2}.$$

Consider the sequence $x_N = (a_{Nm}) \in c_0$. It converges to 0, so that there exists $M \in \mathbb{N}$ such that for any $m \geqslant M$ we have

$$|a_{Nm}| < \frac{\varepsilon}{2}.$$

Therefore, for $m \geqslant M$, we get

$$|a_m| = |a_m - a_{Nm} + a_{Nm}| \leqslant |a_m - a_{Nm}| + |a_{Nm}| < \varepsilon.$$

Hence $x = (a_m) \longrightarrow 0$.

Since c_0 is closed and ℓ^∞ is Banach, c_0 is Banach. □

6. Prove that the space c_0 of sequences with limit 0 is separable, by finding a Schauder basis for c_0.

[*Hint*: You needn't look too hard.]

Solution. I claim that c_0 has the same Schauder basis at the one given in Example 3.21 for ℓ^p: $\{e_1, e_2, \dots\}$ where $e_n = (0, \dots, 0, 1, 0 \dots)$ with the 1 in the n-th spot.

Take $v = (v_n) \in c_0$, then $(v_n) \longrightarrow 0$. I claim that the series

$$\sum_{n=1}^{\infty} v_n e_n$$

converges to v with respect to the norm on c_0, which is the ℓ^∞-norm:

$$\left\| v - \sum_{n=1}^{m} v_n e_n \right\|_{\ell^\infty} = \|(0, \dots, 0, v_{m+1}, v_{m+2}, v_{m+3}, \dots)\|_{\ell^\infty} = \sup_{n \geq m+1} |v_n|,$$

and the latter converges to 0 as $m \longrightarrow \infty$, since $(v_n) \longrightarrow 0$. The uniqueness of the coefficients follows in precisely the same way as for Example 3.21. □

7. Consider the space ℓ^∞ of bounded sequences.
   (a) Let $S \subseteq \ell^\infty$ be the subset of sequences (a_n) such that $a_n \in \{0,1\}$ for all $n \in \mathbb{N}$. Prove that S is an uncountable set.

   [*Hint*: Mimic Cantor's diagonal argument.]

   (b) Use S to construct an uncountable set T of disjoint open balls in ℓ^∞.
   (c) Conclude that ℓ^∞ is not separable.

Solution.

(a) Suppose S is countable and enumerate its elements:

$$\begin{aligned} a_1 &= \big(a_{11}, a_{12}, a_{13}, \dots\big) \\ a_2 &= \big(a_{21}, a_{22}, a_{23}, \dots\big) \\ a_3 &= \big(a_{31}, a_{32}, a_{33}, \dots\big) \\ &\vdots \end{aligned}$$

Go down the diagonal of this infinite grid of 0's and 1's, and define $b_n = 1 - a_{nn}$ for all $n \in \mathbb{N}$. Then $b = (b_n) \in S$, but $b \neq a_m$ for any $m \in \mathbb{N}$, contradiction.

(b) If $a = (a_n), b = (b_n) \in S$ with $a \neq b$ then

$$\|a - b\| = \sup_n |a_n - b_n| = 1,$$

so $\mathbb{B}_{1/2}(a) \cap \mathbb{B}_{1/2}(b) = \varnothing$.

Therefore we can take

$$T = \big\{\mathbb{B}_{1/2}(s) : s \in S\big\}.$$

(c) Any dense subset D of ℓ^∞ must contain at least one point (in fact, must be dense) in each open ball in the set T. Since T is uncountable, D must also be uncountable, so ℓ^∞ is not separable. □