Tutorial Week 07

Topics: metric properties of normed spaces

1. Let $(V, \|\cdot\|)$ be a normed space and let $S \subseteq V$ be a subset. Then $\overline{\mathrm{Span}(S)}$ is the smallest closed subspace of V that contains S.

   Solution. We know that $\mathrm{Span}(S)$ is a subspace of V, and by Example 3.11 that $\overline{\mathrm{Span}(S)}$ is a closed subspace of V.

   Let $W \subseteq V$ be some closed subspace of V that contains S. Then $\mathrm{Span}(S) \subseteq W$, and so $\overline{\mathrm{Span}(S)} \subseteq \overline{W} = W$, whence the minimality property. □

2. Let $(V, \|\cdot\|)$ be a normed space and take $r, s > 0$, $u, v \in V$, $\alpha \in \mathbb{F}^\times$. Show that
   (a) $\mathbb{B}_r(u+v) = \mathbb{B}_r(u) + \{v\}$;
   (b) $\alpha\,\mathbb{B}_1(0) = \mathbb{B}_{|\alpha|}(0)$;
   (c) $\mathbb{B}_r(v) = r\mathbb{B}_1(0) + \{v\}$;
   (d) $r\mathbb{B}_1(0) + s\mathbb{B}_1(0) = (r+s)\mathbb{B}_1(0)$;
   (e) $\mathbb{B}_r(u) + \mathbb{B}_s(v) = \mathbb{B}_{r+s}(u+v)$;
   (f) $\mathbb{B}_1(0)$ is a convex subset of V;
   (g) any open ball in V is convex.

   Solution.

   (a)
$$\begin{aligned} w \in \mathbb{B}_r(u+v) &\iff \|(u+v) - w\| < r \\ &\iff \|u - (w-v)\| < r \\ &\iff w - v \in \mathbb{B}_r(u) \\ &\iff w \in \mathbb{B}_r(u) + \{v\}. \end{aligned}$$

   (b)
$$\begin{aligned} w \in \alpha\mathbb{B}_1(0) &\iff \frac{1}{\alpha} w \in \mathbb{B}_1(0) \\ &\iff \left\|\frac{1}{\alpha} w\right\| < 1 \\ &\iff \|w\| < |\alpha| \\ &\iff w \in \mathbb{B}_{|\alpha|}(0). \end{aligned}$$

   (c) From (a) and (b):
$$\mathbb{B}_r(v) = \mathbb{B}_r(0) + \{v\} = r\mathbb{B}_1(0) + \{v\}.$$

   (d) If $\|u\| < r$ and $\|v\| < s$ then $\|u+v\| < r+s$, so $r\mathbb{B}_1(0) + s\mathbb{B}_1(0) \subseteq (r+s)\mathbb{B}_1(0)$.
   Conversely, if $\|w\| < r+s$, then
$$w = \frac{r}{r+s} w + \frac{s}{r+s} w \in r\mathbb{B}_1(0) + s\mathbb{B}_1(0).$$

(e) From (c) and (d):

$$\mathbb{B}_r(u) + \mathbb{B}_s(v) = r\mathbb{B}_1(0) + s\mathbb{B}_1(0) + \{u\} + \{v\} = (r+s)\mathbb{B}_1(0) + \{u+v\} = \mathbb{B}_{r+s}(u+v).$$

(f) If $u, v \in \mathbb{B}_1(0)$ and $0 \leqslant a \leqslant 1$, then by (d)

$$au + (1-a)v \in a\mathbb{B}_1(0) + (1-a)\mathbb{B}_1(0) = (a+1-a)\mathbb{B}_1(0) = \mathbb{B}_1(0).$$

(g) $\mathbb{B}_r(u) = r\mathbb{B}_1(0) + \{u\}$ is the translate of a convex set, hence is itself convex. □

3. Let $(V, \|\cdot\|)$ be a normed space and let S, T be subsets of V and $\alpha \in \mathbb{F}$. Prove that
   (a) If S and T are bounded, so are $S+T$ and αS.
   (b) If S and T are totally bounded, so are $S+T$ and αS.
   (c) If S and T are compact, so are $S+T$ and αS.

Solution.

(a) A subset S of V is bounded if and only if $S \subseteq \mathbb{B}_s(0) = s\mathbb{B}_1(0)$ for some $s \geqslant 0$. So $S \subseteq s\mathbb{B}_1(0)$ and $T \subseteq t\mathbb{B}_1(0)$, hence $S + T \subseteq s\mathbb{B}_1(0) + t\mathbb{B}_1(0) = (s+t)\mathbb{B}_1(0)$.

Similarly $\alpha S \subseteq s\alpha\mathbb{B}_1(0) = s\mathbb{B}_{|\alpha|}(0) = (s|\alpha|)\mathbb{B}_1(0)$.

(b) Let $\varepsilon > 0$. Since S and T are totally bounded, they can each be covered by finitely many open balls of radius $\varepsilon/2$:

$$S \subseteq \bigcup_{n=1}^{N} \mathbb{B}_{\varepsilon/2}(s_n)$$
$$T \subseteq \bigcup_{m=1}^{M} \mathbb{B}_{\varepsilon/2}(t_m),$$

but then

$$S + T \subseteq \bigcup_{n=1}^{N} \mathbb{B}_{\varepsilon/2}(s_n) + \bigcup_{m=1}^{M} \mathbb{B}_{\varepsilon/2}(t_m) = \bigcup_{n=1}^{N}\bigcup_{m=1}^{M} \big(\mathbb{B}_{\varepsilon/2}(s_n) + \mathbb{B}_{\varepsilon/2}(t_m)\big) = \bigcup_{n=1}^{N}\bigcup_{m=1}^{M} \mathbb{B}_{\varepsilon}(s_n + t_m).$$

For αS, note that S can be covered by finitely many open balls of radius $\varepsilon/|\alpha|$:

$$S \subseteq \bigcup_{n=1}^{N} \mathbb{B}_{\varepsilon/|\alpha|}(s_n),$$

so that

$$\alpha S \subseteq \bigcup_{n=1}^{N} \alpha\mathbb{B}_{\varepsilon/|\alpha|}(s_n) = \bigcup_{n=1}^{N} \mathbb{B}_{\varepsilon}(s_n).$$

(c) Consider the addition map $a\colon V \times V \longrightarrow V$, $a(v,w) = v + w$. We know that it is continuous, so its restriction

$$a|_{S\times T}\colon S \times T \longrightarrow V, \qquad a(s,t) = s + t$$

is also continuous, and its image is $S+T$. Since S and T are compact, so is $S \times T$, and so is $S + T = a(S \times T)$.

The same argument with scalar multiplication gives compactness of αS. □

4. Let $f \in B(V, W)$.
   (a) If U is a subspace of V, then its image $f(U)$ is a subspace of W.
   (b) If U is a closed subspace of W, then its preimage $f^{-1}(U)$ is a closed subspace of V.
   (c) If S is a convex subset of V, then its image $f(S)$ is a convex subset of W.
   (d) If S is a convex subset of W, then its preimage $f^{-1}(S)$ is a convex subset of V.

   Solution.
   (a) Clear since f is linear so it takes vector subspaces to vector subspaces.
   (b) Clear since f is linear so the inverse image of a subspace is a subspace; and f is continuous so the inverse image of a closed set is a closed set.
   (c) Let $f(s), f(t) \in f(S)$ and let $a, b \geqslant 0$ such that $a + b = 1$. We have

   $$af(s) + bf(t) = f(as + bt) \in f(S),$$

   where we used the convexity of S to conclude that $as + bt \in S$.
   (d) Let $u, v \in f^{-1}(S)$ and let $a, b \geqslant 0$ such that $a + b = 1$. Then

   $$f(au + bv) = af(u) + bf(v) \in S,$$

   where we used the convexity of S. We conclude that $au + bv \in f^{-1}(S)$. □

5. Prove that the following subset is a closed subspace of ℓ^1:

   $$S = \left\{ (a_n) \in \ell^1 : \sum_{n=1}^{\infty} a_n = 0 \right\}.$$

   Solution. Consider the function $f\colon \ell^1 \longrightarrow \mathbb{F}$ given by

   $$f\big((a_n)\big) = \sum_{n=1}^{\infty} a_n.$$

   First note that this is a reasonable definition, because the infinite series on the right hand side converges in $\mathbb{F}$:

   $$\left| \sum_{n=1}^{N} a_n \right| \leqslant \sum_{n=1}^{N} |a_n|,$$

   and the latter converges as $N \longrightarrow \infty$ since $(a_n) \in \ell^1$.
   The function f is linear. It is also bounded, because as we have just seen:

   $$\left| f\big((a_n)\big) \right| = \left| \sum_{n=1}^{\infty} a_n \right| \leqslant \sum_{n=1}^{\infty} |a_n| = \|(a_n)\|_{\ell^1}.$$

   Hence $f \in B(\ell^1, \mathbb{F}) = (\ell^1)^\vee$ and its kernel is S, so S is a closed subspace of ℓ^1. □

6. Suppose $1 \leqslant p \leqslant q$. Prove that

   $$\ell^p \subseteq \ell^q.$$

   Show that if $p < q$ then the inclusion is strict: $\ell^p \subsetneq \ell^q$.

Solution. We prove that

$$\|x\|_{\ell^q} \leqslant \|x\|_{\ell^p} \qquad \text{for all } x \in \ell^p.$$

If $\|x\|_{\ell^p} = 0$ then $x = 0$ so $\|x\|_{\ell^q} = 0$ and the inequality obviously holds. So suppose $x \neq 0$, then by dividing through by $\|x\|_{\ell^p}$ we can reduce to proving that

$$\|x\|_{\ell^q} \leqslant 1 \qquad \text{for all } x \text{ such that } \|x\|_{\ell^p} = 1.$$

But if $\|x\|_{\ell^p} = 1$ then

$$\sum_{n=1}^{\infty} |x_n|^p = 1,$$

which means that for all $n \in \mathbb{N}$ we have $|x_n|^p \leqslant 1$, so $|x_n| \leqslant 1$. However, $p \leqslant q$ and $|x_n| \leqslant 1$ implies that $|x_n|^q \leqslant |x_n|^p$ for all $n \in \mathbb{N}$, so that

$$\|x\|_{\ell^q}^q = \sum_{n=1}^{\infty} |x_n|^q \leqslant \sum_{n=1}^{\infty} |x_n|^p = 1.$$

If $p < q$ then $\alpha := q/p > 1$. For each $n \in \mathbb{N}$, let

$$x_n = \frac{1}{n^{1/p}},$$

so that

$$|x_n|^p = \frac{1}{n}, \qquad |x_n|^q = \frac{1}{n^\alpha}.$$

We have

$$\|(x_n)\|_{\ell^p} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty, \qquad \|(x_n)\|_{\ell^q} = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty,$$

so $(x_n) \in \ell^q \setminus \ell^p$. $\square$