Tutorial Week 06

Topics: compact sets, normed spaces, inequalities galore.

1. Let (X, d_X) and (Y, d_Y) be metric spaces, and let d be any conserving metric on $X \times Y$.
   (a) Prove that if X and Y are compact, then $X \times Y$ is compact.
   [*Hint*: If you're not sure where to start, try sequential compactness.]
   (b) Does the converse hold?

   Solution.
   (a) Suppose (x_n, y_n) is a sequence in $X \times Y$. Then (x_n) is a sequence in X, and since X is compact, it follows that (x_n) has some converging subsequence $\left(x_{n_k}\right) \longrightarrow x \in X$. Now consider the sequence $\left(y_{n_k}\right)$ in Y. Since Y is compact, it follows that $\left(y_{n_k}\right)$ has some converging subsequence $\left(y_{n_{k_j}}\right) \longrightarrow y \in Y$. Then $\left(x_{n_{k_j}}\right)$ is a subsequence of the converging sequence $\left(x_{n_k}\right) \longrightarrow x \in X$, so that it is itself converging to $x \in X$. We conclude that $\left(x_{n_{k_j}}, y_{n_{k_j}}\right) \longrightarrow (x, y) \in X \times Y$ and is a subsequence of the original sequence (x_n, y_n).
   (b) The converse does hold, since the projection maps $\pi_1 \colon X \times Y \longrightarrow X$, $\pi_1(x, y) = x$, and $\pi_2 \colon X \times Y \longrightarrow Y$, $\pi_2(x, y) = y$, are continuous and surjective. □

2. Let C be a nonempty compact subset of a metric space (X, d). Prove that there exist points $a, b \in C$ such that
$$d(a, b) = \sup\left\{d(x, y) \colon x, y \in C\right\}.$$
   In other words, the diameter of C is realised as the distance between two points of C.

   Solution. As you know from Assignment 1 Question 5, the distance function $d \colon X \times X \longrightarrow \mathbb{R}$ is continuous. By Q1, $C \times C$ is compact, so by Example 2.73 there exists $(a_{\max}, b_{\max}) \in C \times C$ such that
$$d(a, b) \leqslant d(a_{\max}, b_{\max}) \qquad \text{for all } (a, b) \in C \times C.$$
   Therefore $a_{\max}, b_{\max} \in C$ realise the diameter of C. □

3. A subset S of a vector space V over $\mathbb{F}$ is said to be *convex* if for all $v, w \in S$ and all $a, b \in \mathbb{R}_{\geqslant 0}$ such that $a + b = 1$, we have
$$av + bw \in S.$$
   (In other words, for any two points in S, the line segment joining the two points is entirely contained in S.)
   Show that:
   (a) Any subspace W of V is convex.
   (b) The intersection of an arbitrary collection of convex sets is convex.
   (c) Any interval $I \subseteq \mathbb{R}$ is convex.

   Solution.
   (a) Suppose $v, w \in W$, $a, b \in \mathbb{R}_{\geqslant 0}$ such that $a + b = 1$. In particular, $a, b \in \mathbb{F}$ so $av + bw$ is an $\mathbb{F}$-linear combination of elements of W. Since W is a subspace, $av + bw \in W$.
   (b) Suppose I is an arbitrary set and S_i is a convex subset of V for all $i \in I$. Let
$$S = \bigcap_{i \in I} S_i$$
   and let $v, w \in S$, $a, b \in \mathbb{R}_{\geqslant 0}$ such that $a + b = 1$. Then for all $i \in I$ we have $v, w \in S_i$, so that $av + bw \in S_i$ since S_i is convex. Therefore $av + bw \in S$.

(c) Let $I \subseteq \mathbb{R}$ be an interval and let $v, w \in I$, $a, b \in \mathbb{R}_{\geqslant 0}$ such that $a + b = 1$.

Without loss of generality, $v \leqslant w$. Then

$$av + bw - v = (a-1)v + bw = b(w - v) \geqslant 0 \Rightarrow v \leqslant av + bw$$

and

$$av + bw - w = av + (b-1)w = a(v - w) \leqslant 0 \Rightarrow av + bw \leqslant w.$$

Therefore $v \leqslant av + bw \leqslant w$, hence $av + bw \in I$ by the definition of an interval. □

4. If V is a vector space over $\mathbb{F}$ and $S \subseteq V$ is a convex set, we say that a function $f\colon S \longrightarrow \mathbb{R}$ is *convex* if for all $v, w \in S$ and all $a, b \in \mathbb{R}_{\geqslant 0}$ such that $a + b = 1$, we have

$$f(av + bw) \leqslant af(v) + bf(w).$$

Prove that, if $(V, \|\cdot\|)$ is a normed space, then $f\colon V \longrightarrow \mathbb{R}$ given by $f(v) = \|v\|$ is a convex function.

Solution. Suppose $v, w \in S$ and $a, b \in \mathbb{R}_{\geqslant 0}$ such that $a + b = 1$. Then

$$f(av + bw) = \|av + bw\| \leqslant \|av\| + \|bw\| = |a|\,\|v\| + |b|\,\|w\| = a\|v\| + b\|w\| = af(v) + bf(w).$$ □

5. (a) Prove that the functions

$$\begin{array}{lll} \text{(i)} & f\colon (0,\infty) \longrightarrow \mathbb{R}, & f(x) = x^p, \quad p \geqslant 1 \text{ fixed}, \\ \text{(ii)} & \exp\colon \mathbb{R} \longrightarrow \mathbb{R}, & \exp(x) = e^x, \end{array}$$

are convex.

[*Hint*: Use the second-derivative criterion from Q7.]

(b) Conclude that for any $p \geqslant 1$, any $x, y \geqslant 0$ and any $a, b \geqslant 0$ such that $a + b = 1$, we have

$$(ax + by)^p \leqslant ax^p + by^p.$$

(c) Conclude that for any $x, y \geqslant 0$ and any $a, b \geqslant 0$ such that $a + b = 1$, we have

$$x^a y^b \leqslant ax + by.$$

[*Hint*: Set $x = e^s$, $y = e^t$.]

(d) Show that for any $p \geqslant 1$ and any $x, y \geqslant 0$, we have

$$x^p + y^p \leqslant (x + y)^p.$$

[*Hint*: Let $t = x/y$ and compare derivatives to show that $t^p + 1 \leqslant (t+1)^p$.]

Solution.

(a) (i) We have $f''(x) = p(p-1)x^{p-2} \geqslant 0$ for all $x > 0$, as $p \geqslant 1$.
(ii) We have $\exp''(x) = e^x \geqslant 0$ for all $x \in \mathbb{R}$.

(b) This is exactly the definition of $x \longmapsto x^p$ being a convex function.

(c) If $x = 0$ or $y = 0$, the inequality is trivial, so we may assume $x, y > 0$. Setting $x = e^s$, $y = e^t$, we are trying to prove that

$$e^{as+bt} \leqslant ae^s + be^t,$$

which is the same as e^x being a convex function.

(d) If $y = 0$, the inequality is obvious, so we may assume $y > 0$. Setting $t = x/y$, we are trying to show that
$$t^p + 1 \leqslant (t+1)^p \qquad \text{for all } t \geqslant 0.$$
Let $f\colon \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}$ be given by $f(t) = t^p + 1$, and $g(t)\colon \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}$ be given by $g(t) = (t+1)^p$. We have $f(0) = g(0) = 1$. Also
$$f'(t) = pt^{p-1} \leqslant p(t+1)^{p-1} = g'(t) \qquad \text{for all } t > 0,$$
therefore $f(t) \leqslant g(t)$ for all $t \geqslant 0$, as desired. (There's an appeal to the Mean Value Theorem hiding in here, if you want to write out all the details.) □

6. Let $p \geqslant 1$, $q > 0$, $x, y \geqslant 0$, and $a, b \geqslant 0$ such that $a + b = 1$.

   Prove that
$$\begin{aligned}\min\{x,y\} &\leqslant \left(ax^{-q} + by^{-q}\right)^{-1/q}\\ &\leqslant x^a y^b\\ &\leqslant \left(ax^{1/p} + by^{1/p}\right)^p\\ &\leqslant ax + by\\ &\leqslant \left(ax^p + by^p\right)^{1/p}\\ &\leqslant \max\{x,y\}.\end{aligned}$$

Solution. Without loss of generality $x \leqslant y$ so $\min\{x,y\} = x$ and $\max\{x,y\} = y$.

(a) $x \leqslant y$ so $x^{-1} \geqslant y^{-1}$ so $x^{-q} \geqslant y^{-q}$ so $bx^{-q} \geqslant by^{-q}$ so $ax^{-q} + bx^{-q} \geqslant ax^{-q} + by^{-q}$ so
$$\min\{x,y\} = x = \left(ax^{-q} + bx^{-q}\right)^{-1/q} \leqslant \left(ax^{-q} + by^{-q}\right)^{-1/q}.$$

(b) Let $X = x^{-q}$, $Y = y^{-q}$, then by Q5 part (c) we have
$$X^aY^b \leqslant aX + bY \Rightarrow x^{-aq}y^{-bq} \leqslant ax^{-q} + by^{-q} \Rightarrow x^{aq}y^{bq} \geqslant \left(ax^{-q} + by^{-q}\right)^{-1} \Rightarrow \left(ax^{-q} + by^{-q}\right)^{-1/q} \leqslant x^a y^b.$$

(c) Similar to (b), use Q5 part (c) with $X = x^{1/p}$, $Y = y^{1/p}$.

(d) Use Q5 part (b) with $X = x^{1/p}$, $Y = y^{1/p}$.

(e) Precisely Q5 part (b).

(f) Similar to (a). □

7. Let $I \subseteq \mathbb{R}$ be an interval and let $f\colon I \longrightarrow \mathbb{R}$ be a twice-differentiable function.

   The aim of this Exercise is to check the familiar calculus fact: f is convex if and only if $f''(x) \geqslant 0$ for all $x \in I$.

   It was heavily inspired by Alexander Nagel's Wisconsin notes:

   https://people.math.wisc.edu/~ajnagel/convexity.pdf

   (a) For any $s, t \in I$ with $s < t$, define the linear function $L_{s,t}\colon [s,t] \longrightarrow \mathbb{R}$ by
$$L_{s,t}(x) = f(s) + \left(\frac{x-s}{t-s}\right)\big(f(t) - f(s)\big).$$
   Convince yourself that this is the equation of the secant line joining $\big(s, f(s)\big)$ to $\big(t, f(t)\big)$.

   Prove that f is convex on I if any only if
$$f(x) \leqslant L_{s,t}(x) \qquad \text{for all } s, t \in I \text{ such that } s < t \text{ and all } s \leqslant x \leqslant t.$$

(b) Check that for all $s, t \in I$ such that $s < t$ we have

$$L_{s,t}(x) - f(x) = \frac{x-s}{t-s}\big(f(t) - f(x)\big) - \frac{t-x}{t-s}\big(f(x) - f(s)\big).$$

(c) Use the Mean Value Theorem for f twice to prove that there exist ξ, ζ with $x < \xi < t$ and $s < \zeta < x$ such that

$$L_{s,t}(x) - f(x) = \frac{(t-x)(x-s)}{t-s}\big(f'(\xi) - f'(\zeta)\big).$$

(d) Use the Mean Value Theorem once more to conclude that if $f''(x) \geqslant 0$ for all $x \in I$, then f is convex on I.

(e) Now we prove the converse. From this point on, assume that $f\colon I \longrightarrow \mathbb{R}$ is twice-differentiable and convex, and let $s, t \in I^\circ$.

1. Show that if $s < x < t$ then
$$\frac{f(x) - f(s)}{x-s} \leqslant \frac{f(t) - f(x)}{t-x}.$$
2. Conclude that if $s < x_1 < x_2 < t$ then
$$\frac{f(x_1) - f(s)}{x_1 - s} \leqslant \frac{f(t) - f(x_2)}{t - x_2}.$$
3. Conclude that if $s < t$ then $f'(s) \leqslant f'(t)$, and finally that $f''(x) \geqslant 0$ on I.

Solution. Parts (b)–(d) are pretty thoroughly discussed in the above reference if you need more guidance, so I'll just do parts (a) and (e).

(a) In the definition of convex function, take $v = s$, $w = t$, $a = (t-x)/(t-s)$, $b = (x-s)/(t-s)$, so that $av + bw = x$. Then we know that

$$f(x) \leqslant \frac{t-x}{t-s} f(s) + \frac{x-s}{t-s} f(t) = f(s) + \frac{x-s}{t-s}\big(f(t) - f(s)\big) = L_{s,t}(x).$$

The other direction is straightforward.

(e) 1. From part (a) we have
$$\frac{f(x) - f(s)}{x-s} \leqslant \frac{f(t) - f(s)}{t-s}.$$
Cross-multiplying, we end up with
$$x\big(f(t) - f(s)\big) - s\big(f(t) - f(x)\big) - t\big(f(x) - f(s)\big) \geqslant 0,$$
which is also equivalent to the inequality we are trying to prove.

2. Apply the previous part twice, first with $s < x_1 < x_2$ and then with $x_1 < x_2 < t$, to get
$$\frac{f(x_1) - f(s)}{x_1 - s} \leqslant \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leqslant \frac{f(t) - f(x_2)}{t - x_2}.$$

3. Following from the previous part, we have
$$f'(s) = \lim_{x_1 \searrow s} \frac{f(x_1) - f(s)}{x_1 - s} \leqslant \lim_{x_2 \nearrow t} \frac{f(t) - f(x_2)}{t - x_2} = f'(t).$$
This implies that f' is an increasing function on I°, therefore $f''(x) \geqslant 0$ on I°. □