Tutorial Week 05

Topics: connected, bounded, compact sets.

1. Let A and C be connected subsets of a metric space (X, d). Prove that if $A \cap C \neq \varnothing$, then $A \cup C$ is connected.

   Solution. Suppose $A \cup C$ is disconnected, so that $A \cup C = U \cup V$ with U, V nonempty, disjoint, and open in $A \cup C$.

   Then $A = \big(A \cap U\big) \cup \big(A \cap V\big)$, with $A \cap U, A \cap V$ disjoint and open in A. As A is connected, $A \cap U$ or $A \cap V$ must be empty. Without loss of generality, say $A \cap U = \varnothing$, so that $A \subseteq V$.

   We can apply the same argument to C and get that $C \cap U$ or $C \cap V$ is empty. Since $A \cap C \neq \varnothing$, it must be that $C \cap U = \varnothing$ and $C \subseteq V$.

   But then $U \cup V = A \cup C \subseteq V$, implying that $U \subseteq V$, contradicting the fact that $U \cap V = \varnothing$ and $U \neq \varnothing$. □

2. Let (X, d) be a metric space. Suppose $A \subseteq X$ is a connected subset and $\{C_i : i \in I\}$ is an arbitrary collection of connected subsets of X such that $A \cap C_i \neq \varnothing$ for all $i \in I$. Then

   $$A \cup \bigcup_{i \in I} C_i$$

   is a connected subset of X.

   [*Hint*: Use the argument from Q1.]

   Solution. Suppose

   $$A \cup \bigcup_{i \in I} C_i = U \cup V,$$

   with U, V nonempty disjoint open sets. Then $A = \big(A \cap U\big) \cup \big(A \cap V\big)$, but A is connected so one of these intersections must be empty, say $A \cap U = \varnothing$.

   But $U \subseteq A \cup \bigcup_{i \in I} C_i$, so there must be some $i \in I$ such that $U \cap C_i \neq \varnothing$ (otherwise $U = \varnothing$, contradiction). Since $C_i = \big(C_i \cap U\big) \cup \big(C_i \cap V\big)$ and C_i is connected, we must have $C_i \cap V = \varnothing$, therefore $C_i \subseteq U$.

   But this forces $A \cap C_i \subseteq A \cap U = \varnothing$, contradicting $A \cap C_i \neq \varnothing$ for all $i \in I$. □

3. Let (X, d) be a metric space. Suppose $\{C_n : n \in \mathbb{N}\}$ is a countable collection of connected subsets of X such that $C_n \cap C_{n+1} \neq \varnothing$ for all $n \in \mathbb{N}$. Then

   $$\bigcup_{n \in \mathbb{N}} C_n$$

   is a connected subset of X.

   [*Hint*: Build the union inductively, and use Q1 and Q2.]

   Solution. For any $N \in \mathbb{N}$, let

   $$A_N = \bigcup_{n=1}^{N} C_n.$$

   We use induction to prove that A_N is connected for all $N \in \mathbb{N}$.

   The base case $N = 1$ is clear as $A_1 = C_1$.

For the induction step, fix $N \in \mathbb{N}$ and suppose A_N is connected. Then $A_{N+1} = A_N \cup C_{N+1}$ is connected by Q1.

So $\{A_N : N \in \mathbb{N}\}$ is a collection of connected sets, and A_1 is a connected set such that $A_1 \cap A_N \neq \varnothing$ for all $N \in \mathbb{N}$. By Q2,

$$A_1 \cup \bigcup_{N\in\mathbb{N}} A_N = \bigcup_{N\in\mathbb{N}} A_N = \bigcup_{n\in\mathbb{N}} C_n$$

is connected. □

4. Let (X, d) be a metric space and define $x \sim x'$ if there exists a connected subset $C \subset X$ such that $x, x' \in C$.

   Prove that this is an equivalence relation on the set X thereby partitioning X into a disjoint union of maximal connected subsets (these are called the *connected components* of X).

   [*Hint*: Recall that an equivalence relation has three defining axioms: (a) $x \sim x$ for all $x \in X$; (b) if $x \sim x'$ then $x' \sim x$; (c) if $x \sim x'$ and $x' \sim x''$ then $x \sim x''$.]

   Solution.

   (a) $x \sim x$: for any $x \in X$, the set $C = \{x\}$ is connected and contains x, so $x \sim x$.

   (b) if $x \sim x'$ then $x' \sim x$: clear from the definition, which does not distinguish x and x'.

   (c) if $x \sim x'$ and $x' \sim x''$ then $x \sim x''$: since $x \sim x'$ there exists a connected set C_1 such that $x, x' \in C_1$; since $x' \sim x''$ there exists a connected set C_2 such that $x', x'' \in C_2$; by Q1, since C_1 and C_2 are connected and $x' \in C_1 \cap C_2$, the union $C_1 \cup C_2$ is connected, and it contains both x and x'', so that $x \sim x''$.

   □

5. Give explicit continuous surjective functions $f : \mathbb{R} \longrightarrow I$, where I is:

   (a) $\mathbb{R}$ (b) $(0, \infty)$ (c) $(-\infty, 0)$ (d) $(-\infty, 0]$ (e) $[-1, 1]$
   (f) $(0, 1]$ (g) $[0, 1)$ (h) $(-\pi/2, \pi/2)$ (i) $\{0\}$.

   [*Hint*: Draw some functions you know from calculus and see what their ranges are.]

   Solution. These are of course not the only possible answers (well, except for the last one).

   (a) $x \longmapsto x$;

   (b) $x \longmapsto e^x$;

   (c) $x \longmapsto -e^x$;

   (d) $x \longmapsto -x^2$;

   (e) $x \longmapsto \sin(x)$;

   (f) $x \longmapsto \min\{e^x, 1\}$;

   (g) $x \longmapsto \max\{-e^x, -1\} + 1$;

   (h) $x \longmapsto \arctan(x)$;

   (i) $x \longmapsto 0$.

   □

6. Let (X, d) be a metric space.

   If A and B are bounded sets with $A \cap B \neq \varnothing$, then
   $$\operatorname{diam}(A \cup B) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B).$$

   Solution. It suffices to show that for any $x, y \in A \cup B$ we have
   $$d(x, y) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B).$$
   If $x, y \in A$, this is obvious as $d(x, y) \leqslant \operatorname{diam}(A)$. Similarly if $x, y \in B$.

   It remains to see what happens if $x \in A$ and $y \in B$. Let $t \in A \cap B$. We have
   $$d(x, y) \leqslant d(x, t) + d(t, y) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B),$$
   as desired. □

7. Let C be a closed subset of a compact subset K of a metric space (X, d). Prove that C is compact.

   [*Hint*: $K \subseteq X = C \cup \big(X \setminus C\big)$.]

   Solution. Consider an arbitrary open cover of C:
   $$C \subseteq \bigcup_{i \in I} U_i.$$
   Then we have
   $$K \subseteq X = C \cup \big(X \setminus C\big) \subseteq \left(\bigcup_{i \in I} U_i\right) \cup \big(X \setminus C\big),$$
   which is an open cover of K. As K is compact, there is a finite subcover, so that
   $$K \subseteq \left(\bigcup_{n=1}^{N} U_{i_n}\right) \cup \big(X \setminus C\big), \qquad i_n \in I,$$
   hence
   $$C \subseteq \bigcup_{n=1}^{N} U_{i_n}.$$
   □

8. Let K and L be compact subsets of a metric space (X, d). Prove that $K \cup L$ is compact.

   Solution. Consider an arbitrary open cover of $K \cup L$:
   $$K \cup L \subseteq \bigcup_{i \in I} U_i.$$
   This is also an open cover of K, so there is a finite subcover that still covers K:
   $$K \subseteq \bigcup_{n=1}^{N} U_{i_n}, \qquad i_n \in I.$$
   Similarly, we get a finite subcover that covers L:
   $$L \subseteq \bigcup_{m=1}^{M} U_{j_m}, \qquad j_m \in I.$$
   Letting $S = \{i_1, \ldots, i_N\} \cup \{j_1, \ldots, j_M\}$, we get a finite subcover that covers $K \cup L$:
   $$K \cup L \subseteq \bigcup_{s \in S} U_s.$$
   □