Tutorial Week 04

Topics: completeness, uniform continuity.

1. Let (X, d_X) and (Y, d_Y) be metric spaces and let d be a conserving metric on $X \times Y$.
   (a) Prove that the sequence $\big((x_n, y_n)\big)$ is Cauchy in $X \times Y$ if and only if (x_n) is Cauchy in X and (y_n) is Cauchy in Y.
   (b) Prove that if X and Y are complete then $X \times Y$ is complete. Is the converse true?

   Solution.
   (a) Suppose $\big((x_n, y_n)\big)$ is a Cauchy sequence in $(X \times Y, d)$. Fix $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $m, n \geqslant N$ we have

   $$d_X(x_m, x_n) \leqslant \max\big\{d_X(x_m, x_n), d_Y(y_m, y_n)\big\} \leqslant d\big((x_m, y_m), (x_n, y_n)\big) < \varepsilon,$$

   so (x_n) is Cauchy in X. Similarly, (y_n) is Cauchy in Y.

   Conversely, suppose (x_n) is Cauchy in X and (y_n) is Cauchy in Y. Fix $\varepsilon > 0$. Let $N_x \in \mathbb{N}$ be such that for all $m, n \geqslant N_x$ we have $d_X(x_m, x_n) < \varepsilon/2$. Let $N_y \in \mathbb{N}$ be such that for all $m, n \geqslant N_y$ we have $d_Y(y_m, y_n) < \varepsilon/2$. Let $N = \max\{N_x, N_y\}$, then for all $m, n \geqslant N$ we have

   $$d\big((x_m, y_m), (x_n, y_n)\big) \leqslant d_X(x_m, x_n) + d_Y(y_m, y_n) < \varepsilon,$$

   so $\big((x_n, y_n)\big)$ is Cauchy in $X \times Y$.

   (b) Let $\big((x_n, y_n)\big)$ be a Cauchy sequence in $X \times Y$. By part (a), (x_n) is Cauchy in X and (y_n) is Cauchy in Y. Since X and Y are complete, we have $(x_n) \longrightarrow x \in X$ and $(y_n) \longrightarrow y \in Y$. By Exercise 2.22, $\big((x_n, y_n)\big) \longrightarrow (x, y) \in X \times Y$.

   The converse also holds: suppose $X \times Y$ is complete. Let (x_n) be a Cauchy sequence in X, and fix some $y \in Y$. Then by (a) we have that $\big((x_n, y)\big)$ is Cauchy in $X \times Y$, so $\big((x_n, y)\big) \longrightarrow (x, y) \in X \times Y$, which by Exercise 2.22 implies that $(x_n) \longrightarrow x \in X$. The same proof gives us that Y is complete.

   □

2. Any distance-preserving function is uniformly continuous.

   Solution. This is immediate from the definitions (can take $\delta = \varepsilon$). □

3. Check (directly from the definition of uniform continuity) that $f\colon \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ given by $f(x) = \frac{1}{x}$ is not uniformly continuous.

   Solution. First make sure that you negate the condition in the definition correctly: there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $x' \in \mathbb{B}_\delta(x)$ such that $f(x') \notin \mathbb{B}_\varepsilon\big(f(x)\big)$.

   And now, to work: let $\varepsilon = 1$. Take an arbitrary $\delta > 0$. Set $x = \min\{\delta, 1\}$. I claim that $x' := x/2$ satisfies the desired condition. Let's check:

   $$|x - x'| = \frac{x}{2} \leqslant \frac{\delta}{2} < \delta,$$

   so indeed $x' \in \mathbb{B}_\delta(x)$.

   Also

   $$|f(x) - f(x')| = \left|\frac{1}{x} - \frac{1}{x'}\right| = \left|\frac{1}{x} - \frac{2}{x}\right| = \frac{1}{x} \geqslant 1 = \varepsilon,$$

   so indeed $f(x') \notin \mathbb{B}_\varepsilon\big(f(x)\big)$. □

4. Let $f\colon X \longrightarrow Y$ be a uniformly continuous function between two metric spaces and suppose $(x_n) \sim (x'_n)$ are equivalent sequences in X. Prove that $\big(f(x_n)\big) \sim \big(f(x'_n)\big)$ as sequences in Y.

   Does the conclusion hold if f is only assumed to be continuous?

   Solution. Let $\varepsilon > 0$. As f is uniformly continuous, there exists $\delta > 0$ such that for all $x, x' \in X$, if $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$. As $(x_n) \sim (x'_n)$, there exists $N \in \mathbb{N}$ such that $d_X(x_n, x'_n) < \delta$ for all $n \geqslant N$. Hence for all $n \geqslant N$ we have $d_Y(f(x_n), f(x'_n)) < \varepsilon$.

   The result does not hold in general for continuous functions; for instance one can take $f\colon \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ given by $f(x) = \frac{1}{x}$, and $(1/n) \sim (1/n^2)$ but $(f(1/n)) = (n)$, $(f(1/n^2)) = (n^2)$ and $(n) \not\sim (n^2)$. ☐

5. Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \longrightarrow Y$ a surjective continuous function. Suppose that X is complete and for all $x_1, x_2 \in X$ we have

$$d_X\big(x_1, x_2\big) \leqslant d_Y\big(f(x_1), f(x_2)\big).$$

   (a) Prove that Y is complete.

   In particular, distance-preserving maps preserve completeness.

   (b) Do you feel strongly that uniformly continuous functions ought to preserve completeness? (After all, they preserve Cauchy sequences, and completeness is defined in terms of Cauchy sequences.)

   Prove that $f\colon \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$ given by $f(x) = \arctan(x)$ is uniformly continuous, but...

   Solution.

   (a) Let (y_n) be a Cauchy sequence in Y. For each $n \in \mathbb{N}$, let $x_n \in f^{-1}(y_n)$. I claim that (x_n) is a Cauchy sequence in X. Fix $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that for all $m, n \geqslant N$ we have $d_Y(y_m, y_n) < \varepsilon$. Then for all $m, n \geqslant N$ we have

$$d_X(x_m, x_n) \leqslant d_Y\big(f(x_m), f(x_n)\big) = d_Y(y_m, y_n) < \varepsilon,$$

   so (x_n) is indeed Cauchy in X.

   Since X is complete, we have $(x_n) \longrightarrow x \in X$, so that by the continuity of f we conclude that $(y_n) = \big(f(x_n)\big) \longrightarrow f(x) \in Y$.

   (b) Given $x_1 < x_2$, apply the Mean Value Theorem to $f(x) = \arctan(x)$ on $[x_1, x_2]$ to get some $\xi \in (x_1, x_2)$ such that

$$|f(x_2) - f(x_1)| = |f'(\xi)|\,|x_2 - x_1| = \frac{1}{1+\xi^2}\,|x_2 - x_1| \leqslant |x_2 - x_1|.$$

   So for any $\varepsilon > 0$ we can take $\delta = \varepsilon$ and conclude that f is uniformly continuous.

   It is also surjective onto $(-\pi/2, \pi/2)$, but the latter is of course not complete.

   ☐

6. Any Cauchy sequence (x_n) is *bounded*, that is there exists $C \geqslant 0$ such that $d(x_n, x_m) \leqslant C$ for all $n, m \in \mathbb{N}$.

Solution. Let $N \in \mathbb{N}$ be such that for all $m, n \geqslant N$ we have $d(x_m, x_n) < 1$.

Let $B = \max\{d(x_m, x_N) : 1 \leqslant m < N\}$. Let $C = 2B + 1$, then we have

$$d(x_m, x_n) \leqslant \begin{cases} 1 \leqslant C & \text{if } m, n \geqslant N \\ d(x_m, x_N) + d(x_N, x_n) \leqslant B + 1 \leqslant C & \text{if } m < N, n \geqslant N \\ d(x_m, x_N) + d(x_N, x_n) \leqslant 2B \leqslant C & \text{if } m, n < N. \end{cases}$$

□

7. Suppose A and B are abelian groups. A function $f : A \longrightarrow B$ is called *additive* if $f(a + b) = f(a) + f(b)$.

(a) Prove that every additive function $f : A \longrightarrow B$ satisfies

$$f(0) = 0 \qquad \text{and} \qquad f(-a) = -f(a).$$

(b) Let V be a $\mathbb{Q}$-vector space. Prove that every additive function $f : \mathbb{Q} \longrightarrow V$ is $\mathbb{Q}$-linear.

(c) What can you say (and prove) about **continuous** additive functions $\mathbb{R} \longrightarrow \mathbb{R}$?

(d) Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is additive and continuous at 0. Prove that f is continuous on $\mathbb{R}$, and conclude that f is $\mathbb{R}$-linear.

(e) Let B be a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. (Recall from Exercise 1.5 that B is uncountable.) Use two distinct irrational elements of B to construct a $\mathbb{Q}$-linear transformation $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is not $\mathbb{R}$-linear.

If you would (and why wouldn't you?), follow the rabbit:

https://en.wikipedia.org/wiki/Cauchy%27s_functional_equation

Solution.

(a) $f(0) = f(0 + 0) = f(0) + f(0)$ so $f(0) = 0$.

$f(-a) + f(a) = f(-a + a) = f(0) = 0$.

(b) Let $v = f(1) \in V$.

For $n \in \mathbb{N}$ we have

$$f(n) = f(1 + 1 + \cdots + 1) = f(1) + \cdots + f(1) = nv.$$

For $m \in \mathbb{N}$ we have

$$v = f(1) = f\left(\frac{1}{m} + \cdots + \frac{1}{m}\right) = mf\left(\frac{1}{m}\right),$$

so $f(1/m) = (1/m)v$.

Therefore, for any $n, m \in \mathbb{N}$ we have

$$f\left(\frac{n}{m}\right) = nf\left(\frac{1}{m}\right) = \frac{n}{m}v.$$

Combining this with $f(-a) = -f(a)$ and $f(0) = 0$, we conclude that $f(x) = xv = xf(1)$ for all $x \in \mathbb{Q}$.

(c) Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be additive. Let $g\colon \mathbb{Q} \longrightarrow \mathbb{R}$ be the restriction of f to $\mathbb{Q} \subseteq \mathbb{R}$. Let $a = g(1) = f(1)$.

By part (b), $g(q) = q\,g(1) = qa$ for all $q \in \mathbb{Q}$. Let $x \in \mathbb{R}$. As $\mathbb{Q}$ is dense in $\mathbb{R}$, there is some sequence $(q_n) \longrightarrow x$ with $q_n \in \mathbb{Q}$; since f is continuous we have

$$f(x) = f\left(\lim_{n\longrightarrow\infty} q_n\right) = \lim_{n\longrightarrow\infty} f(q_n) = \lim_{n\longrightarrow\infty} g(q_n) = \lim_{n\longrightarrow\infty} (q_n a) = xa = xf(1).$$

Hence f is $\mathbb{R}$-linear.

(d) Let $x \in \mathbb{R}$. Fix $\varepsilon > 0$. Let $\delta > 0$ be such that if $|t| < \delta$, then $|f(t)| < \varepsilon$.

Suppose $x' \in \mathbb{R}$ is such that $|x - x'| < \delta$, then

$$|f(x) - f(x')| = |f(x - x')| < \varepsilon.$$

So f is continuous at every $x \in \mathbb{R}$, so by part (c) f is $\mathbb{R}$-linear.

(e) Let B be a $\mathbb{Q}$-basis for $\mathbb{R}$. Exactly one element of B is a nonzero rational, and without loss of generality we may assume it is 1. Since B is uncountable, it also contains uncountably many irrationals. Let $b, c \in B \cap \big(\mathbb{R} \setminus \mathbb{Q}\big)$. Consider the bijective function $\sigma\colon B \longrightarrow B$ given by

$$\sigma(b) = c, \qquad \sigma(c) = b, \qquad \sigma(x) = x \text{ for all } x \in B \setminus \{b, c\}.$$

Since B is a $\mathbb{Q}$-basis of $\mathbb{R}$, σ extends by $\mathbb{Q}$-linearity to a $\mathbb{Q}$-linear transformation $f\colon \mathbb{R} \longrightarrow \mathbb{R}$. In particular, f is additive.

Suppose that f is $\mathbb{R}$-linear, then:

$$c = f(b) = bf(1) = b1 = b,$$

contradicting the fact that $b \neq c$.

□