Tutorial Week 03

Topics: convergence of sequences, continuous functions, nowhere dense sets, equivalence of metrics

1. Let X and Y be two metric spaces and endow the Cartesian product $X \times Y$ with the Manhattan metric from Example 2.3. Prove that a sequence $\big((x_n, y_n)\big)$ in $X \times Y$ converges to (x, y) if and only if (x_n) converges to x and (y_n) converges to y.

   Solution. By definition,

   $$d\big((x_n, y_n), (x, y)\big) = d_X(x_n, x) + d_Y(y_n, y).$$

   Suppose $(x_n) \longrightarrow x$ and $(y_n) \longrightarrow y$. Let $\varepsilon > 0$, $N_x \in \mathbb{N}$ such that $x_n \in \mathbb{B}_{\varepsilon/2}(x)$ for all $n \geqslant N_x$, and $N_y \in \mathbb{N}$ such that $y_n \in \mathbb{B}_{\varepsilon/2}(y)$ for all $n \geqslant N_y$. Set $N = \max\{N_x, N_y\}$, then

   $$d\big((x_n, y_n), (x, y)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } n \geqslant N.$$

   Conversely, suppose $\big((x_n, y_n)\big) \longrightarrow (x, y)$. Given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $(x_n, y_n) \in \mathbb{B}_\varepsilon\big((x, y)\big)$ for all $n \geqslant N$, therefore

   $$d_X(x_n, x) + d_Y(y_n, y) = d\big((x_n, y_n), (x, y)\big) < \varepsilon.$$

   Since both d_X and d_Y are non-negative, we conclude that each summand is strictly bounded by ε for all $n \geqslant N$. □

2. Let (x_n) be a sequence in X, let $\varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ be an injective function, and consider the sequence $(y_n) = \big(x_{\varphi(n)}\big)$ in X. Prove that if (x_n) converges to x, then so does (y_n).

   Does the converse hold?

   Solution. Suppose $(x_n) \longrightarrow x$. Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $x_n \in \mathbb{B}_\varepsilon(x)$ for all $n \geqslant N$. Since $\varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ is injective, the inverse image $\varphi^{-1}\big(\{1, \dots, N-1\}\big)$ is a finite set, so it has a maximal element M. (If the set is empty, just take $M = 0$.) For all $n \geqslant M + 1$, we have $\varphi(n) \geqslant N$, so $y_n = x_{\varphi(n)} \in \mathbb{B}_\varepsilon(x)$.

   The converse certainly does not hold. For instance, take $(x_n) = (1, 0, 1, 0, 1, 0, \dots)$ and $\varphi(n) = 2n$, then the sequence $(y_n) = (0, 0, 0, \dots)$ converges to 0 but (x_n) does not converge. □

3.

   (a) Let $f\colon X \longrightarrow Y$ be a function between two sets X and Y, and let $S \subseteq Y$. Prove that

   $$f^{-1}(S) = X \setminus f^{-1}(Y \setminus S).$$

   (b) Let $f\colon X \longrightarrow Y$ be a function between metric spaces. Prove that f is continuous if and only if: for any closed subset $C \subseteq Y$, the inverse image $f^{-1}(C) \subseteq X$ is a closed subset.

   Solution.

   (a) We have $x \in f^{-1}(S)$ iff $f(x) \in S$ iff $f(x) \notin \big(Y \setminus S\big)$ iff $x \notin f^{-1}(Y \setminus S)$.

(b) Suppose f is continuous and $C \subseteq Y$ is closed. By part (a) we have

$$f^{-1}(C) = X \setminus f^{-1}(Y \setminus C).$$

Then $\big(Y \setminus C\big) \subseteq Y$ is open, so by Example 2.26, $f^{-1}(Y \setminus C) \subseteq X$ is open, therefore $f^{-1}(C)$ is closed.

Conversely, suppose the inverse image of any closed subset is closed. Let $V \subseteq Y$ be open, then by part (a) we have

$$f^{-1}(V) = X \setminus f^{-1}(Y \setminus V).$$

So $\big(Y \setminus V\big) \subseteq Y$ is closed, so $f^{-1}(Y \setminus V) \subseteq X$ is closed, hence $f^{-1}(V)$ is open. By Example 2.26, f is continuous.

□

4. Show that if $f\colon X \longrightarrow Y$ is a continuous map between metric spaces and $A \subseteq X$ then $f\big(\overline{A}\big) \subseteq \overline{f(A)}$.

Solution. Let $x \in \overline{A}$, let $y = f(x)$, and suppose that $y \notin \overline{f(A)}$. By Exercise 2.9 part (a), there exists an open neighbourhood $V \subseteq \big(Y \setminus f(A)\big)$ with $y \in V$. As f is continuous, there exists an open neighbourhood $U \subseteq X$ of x with $f(U) \subseteq V$; as V does not intersect $f(A)$, we get that U does not intersect A, contradicting the fact that $x \in \overline{A}$. □

5. Give $\mathbb{N}$ the metric induced from $\mathbb{R}$. Let (X, d) be a metric space and (x_n) a sequence in X. Prove that (x_n) is a continuous function $\mathbb{N} \longrightarrow X$.

Solution. First note that the induced metric on $\mathbb{N} \subseteq \mathbb{R}$ is equivalent to the discrete metric: for any $n \in \mathbb{N}$, we have $\{n\} = (n-1, n+1) \cap \mathbb{N}$, so $\{n\}$ is open in $\mathbb{N}$. Therefore every subset of $\mathbb{N}$ is open, hence every function $\mathbb{N} \longrightarrow X$ is continuous. □

6.

(a) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be functions, where X, Y, Z are sets, and let $S \subseteq Z$. Then

$$f^{-1}\big(g^{-1}(S)\big) = (g \circ f)^{-1}(S).$$

(b) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be continuous functions, where X, Y, Z are metric spaces. Prove that $g \circ f\colon X \longrightarrow Z$ is continuous.

Solution.

(a) We have $x \in (g \circ f)^{-1}(S)$ iff $(g \circ f)(x) \in S$ iff $g(f(x)) \in S$ iff $f(x) \in g^{-1}(S)$ iff $x \in f^{-1}\big(g^{-1}(S)\big)$.

(b) Let $W \subseteq Z$ be open. As $g\colon Y \longrightarrow Z$ is continuous, $g^{-1}(W) \subseteq Y$ is open. As $f\colon X \longrightarrow Y$ is continuous, $(g \circ f)^{-1}(W) = f^{-1}\big(g^{-1}(W)\big) \subseteq X$ is open. So $g \circ f$ is continuous.

□

7. Let $f\colon X \longrightarrow Y$ be a continuous map between metric spaces and let $S \subseteq Y$ be such that $f(X) \subseteq S$. Endowing S with the metric induced from Y, show that $f\colon X \longrightarrow S$ is continuous.

Solution. Since $f(X) \subseteq S$, we have that $f^{-1}(Y \setminus S) = \varnothing$.

Let $W \subseteq S$ be open in (the induced metric on) S, then there exists $V \subseteq Y$ open in Y such that $W = V \cap S$. Since $f\colon X \longrightarrow Y$ is continuous, we have that $U := f^{-1}(V)$ is open in X. But $f^{-1}(V) = f^{-1}(V \cap S) \cup f^{-1}(V \setminus S)$, and $f^{-1}(V \setminus S) \subseteq f^{-1}(Y \setminus S) = \varnothing$, so $f^{-1}(V) = f^{-1}(V \cap S) = f^{-1}(W)$ is open in X. □

8. Let $g_1\colon X \longrightarrow Y_1$ and $g_2\colon X \longrightarrow Y_2$ be continuous maps, with X, Y_1, Y_2 metric spaces.

   Define $f\colon X \longrightarrow Y_1 \times Y_2$ by $f(x) = \big(g_1(x), g_2(x)\big)$. Endow $Y_1 \times Y_2$ with the Manhattan metric.

   Show that f is continuous if and only if both g_1 and g_2 are continuous.

   Solution. The function f is continuous iff for any sequence $(x_n) \longrightarrow x \in X$, we have $\big(f(x_n)\big) \longrightarrow f(x) \in Y_1 \times Y_2$, in other words $\big(g_1(x_n), g_2(x_n)\big) \longrightarrow \big(g_1(x), g_2(x)\big) \in Y_1 \times Y_2$. But by Exercise 2.22, the latter holds iff $\big(g_1(x_n)\big) \longrightarrow g_1(x) \in Y_1$ and $\big(g_2(x_n)\big) \longrightarrow g_2(x) \in Y_2$, which precisely says that both g_1 and g_2 are continuous. □

9. If A and B are subsets of a metric space (X, d), then

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

   Solution. By Exercise 2.9 part (b), $A \subseteq A \cup B$ implies $\overline{A} \subseteq \overline{A \cup B}$, and similarly for $\overline{B} \subseteq \overline{A \cup B}$.

   For the other inclusion, note that by Example 2.15, $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, so by the minimality of the closure Exercise 2.9, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. □

10. Let (X, d) be a metric space.

    (a) Prove that any subset of a nowhere dense subset of X is nowhere dense in X.

    (b) Prove that a subset $N \subseteq X$ is nowhere dense if and only if $X \setminus \overline{N}$ is dense in X.

    (c) Prove that the union of any finite collection of nowhere dense subsets of X is nowhere dense in X.

    Solution.

    (a) Let $N \subseteq X$ be nowhere dense and let $M \subseteq N$. Then $\overline{M} \subseteq \overline{N}$ by Exercise 2.9 part (b), so $\big(\overline{M}\big)^\circ \subseteq \big(\overline{N}\big)^\circ = \varnothing$ by Exercise 2.3.

    (b) Suppose N is nowhere dense and let $U \subseteq X$ be nonempty and open. If $U \cap \big(X \setminus \overline{N}\big) = \varnothing$, then $U \subseteq \overline{N}$, so $U \subseteq \big(\overline{N}\big)^\circ = \varnothing$, contradicting the non-emptiness of U. So it must be that U intersects $X \setminus \overline{N}$ nontrivially, hence $X \setminus \overline{N}$ is dense.

    Conversely, suppose $X \setminus \overline{N}$ is dense but N is not nowhere dense, that is there exists a nonempty open $U \subseteq \overline{N}$. Then $U \cap \big(X \setminus \overline{N}\big) = \varnothing$, contradicting the denseness of $X \setminus \overline{N}$.

    (c) It suffices to prove the case of two nowhere dense sets M and N. Let $L = M \cup N$. Then by Exercise 2.30 we have $\overline{L} = \overline{M} \cup \overline{N}$ so $X \setminus \overline{L} = \big(X \setminus \overline{M}\big) \cap \big(X \setminus \overline{N}\big)$. As $X \setminus \overline{L}$ is the union of two dense open subsets, it is dense and open by Exercise 2.12, hence L is nowhere dense.

    □

11. Let X be a set.

(a) Show that the relation "d_1 is finer than d_2" on metrics on X gives rise to a relation "$[d_1]$ is finer than $[d_2]$" on equivalence classes of metrics on X.

(b) Show that the latter is a partial order on the set of equivalence classes of metrics on X.

(c) In the statement from part (b), can we remove the words "equivalence classes of"?

(d) Show that the partial order from part (b) has a unique maximal element.

Solution.

(a) First we note that the relation "is finer than" on metrics is transitive: if d_1 is finer than d_2 and d_2 is finer than d_3 then d_1 is finer than d_3. (This is clear from any of the equivalent definitions in Proposition 2.27.)

Next we show that the relation "is finer than" on equivalence classes of metrics is well-defined. Let $[d]$ denote the equivalence class of a metric d. We say that a class $[d_1]$ is finer than a class $[d_2]$ if the metric d_1 is finer than the metric d_2. To check well-definedness of this concept, suppose that d_1' is a metric equivalent to d_1, and d_2' is a metric equivalent to d_2. Is is true that d_1' is finer than d_2'? Well, d_1' is finer than d_1, which is finer than d_2, which is finer than d_2', so the answer is yes, by transitivity.

(b) Given a class $[d]$, it is true that d is a finer metric than d, so $[d]$ is a finer class than $[d]$.

If $[d_1]$ is a finer class than $[d_2]$ and $[d_2]$ is a finer class than $[d_1]$, then d_1 is a finer metric than d_2 and d_2 is a finer metric than d_1, hence d_1 and d_2 are equivalent metrics, so $[d_1] = [d_2]$.

Finally, suppose $[d_1]$ is a finer class than $[d_2]$, which is a finer class than $[d_3]$. Then d_1 is a finer metric than d_2, which is a finer metric than d_3, so by the transitivity we saw in part (a), d_1 is a finer metric than d_3, so $[d_1]$ is a finer class than $[d_3]$.

(c) Not in general, as for metrics, d_1 finer than d_2 and d_2 finer than d_1 does not necessarily imply that $d_1 = d_2$, only that they are equivalent metrics.

(d) The unique maximal element is the equivalence class of the discrete metric on X, as it is clear that the discrete metric is finer than any metric on X.

□