Tutorial Week 02

Topics: infinite-dimensional vector spaces, metrics, open sets.

1. Let $\mathbb{R}^\infty$ be the set of arbitrary sequences $(x_1, x_2, \dots)$ of elements of $\mathbb{R}$.

   This is a vector space under the naturally-defined addition of sequences and multiplication by a scalar.

   Let $e_j \in \mathbb{R}^\infty$ be the sequence whose j-th entry is 1, and all the others are 0. Describe the subspace $\operatorname{Span}\{e_1, e_2, \dots\}$ of $\mathbb{R}^\infty$. Is the set $\{e_1, e_2, \dots\}$ a basis of $\mathbb{R}^\infty$?

   Solution. Let $S = \{e_1, e_2, \dots\}$ and $W = \operatorname{Span}(S)$.

   For each $n \in \mathbb{N}$, define
   $$W_n = \operatorname{Span}\{e_1, e_2, \dots, e_n\} \subseteq W.$$
   I claim that
   $$W = \bigcup_{n\in\mathbb{N}} W_n.$$
   One inclusion is clear, as $W_n \subseteq W$ for all $n \in \mathbb{N}$.

   For the other inclusion, let $w \in W$. Then there exist $m \in \mathbb{N}$, $a_1, \dots, a_m \in \mathbb{R}$ and $k_1, \dots, k_m \in \mathbb{N}$ such that
   $$w = a_1 e_{k_1} + \cdots + a_m e_{k_m}.$$
   Set $n = \max\{k_1, \dots, k_m\}$, then $w \in W_n$.

   Is $W = \mathbb{R}^\infty$? No. Any $w \in W$ appears in a W_n for some $n \in \mathbb{N}$, therefore only the first n entries of w can be nonzero. This means, for instance, that $v = (1, 1, 1, \dots) \notin W$. So S does not span $\mathbb{R}^\infty$. ☐

2. Let $V = \mathbb{R}$ viewed as a vector space over $\mathbb{Q}$.

   Let $\alpha \in \mathbb{R}$. Show that the set $T = \{\alpha^n : n \in \mathbb{N}\}$ is $\mathbb{Q}$-linearly independent if and only if α is transcendental.

   (Note: An element $\alpha \in \mathbb{R}$ is called algebraic if there exists a monic polynomial $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$. An element $\alpha \in \mathbb{R}$ is called transcendental if it is not algebraic.)

   Solution. This is a straightforward rewriting of the definition of algebraic: α is algebraic if and only if it satisfies a polynomial equation with coefficients in $\mathbb{Q}$, which is equivalent to a nontrivial linear relation between the powers of α, which exists if and only if T is linearly dependent. ☐

3. Let W be a $\mathbb{Q}$-vector space with a countable basis B. Show that W is a countable set.

   [*Hint*: Use Exercise 1.2.]

   Conclude that $\mathbb{R}$ does not have a countable basis as a vector space over $\mathbb{Q}$.

   Solution. Since B is countable we can enumerate it as $B = \{b_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $W_n = \operatorname{Span}\{b_1, \dots, b_n\}$. Then for each $n \in \mathbb{N}$, W_n is isomorphic (as a $\mathbb{Q}$-vector space) to $\mathbb{Q}^n$, hence W_n is countable. I claim that
   $$W = \bigcup_{n\in\mathbb{N}} W_n.$$

One inclusion is obvious, as $W_n \subseteq W$ for all $n \in \mathbb{N}$. For the other direction, let $w \in W = \operatorname{Span}(B)$, so there exist $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{Q}$ and $k_1, \ldots, k_n \in \mathbb{N}$ such that

$$w = a_1 b_{k_1} + \cdots + a_n b_{k_n}.$$

Let $k = \max\{k_1, \ldots, k_n\}$, then $w \in W_k$.

So W is a countable union of countable sets, hence countable by Exercise 1.2.

The last claim follows directly from the fact that $\mathbb{R}$ is an uncountable set. □

4. Let $(X, \leqslant)$ be a nonempty finite poset. (This just means that X is a nonempty finite set with a partial order $\leqslant$.) Prove that X has a maximal element.

   [*Hint*: You could, for instance, use induction on the number of elements of X.]

   Solution. We proceed by induction on n, the cardinality of X.

   Base case: if $n = 1$ then $X = \{x\}$ for a single element x. Then trivially x is a maximal element of X.

   For the induction step, fix $n \in \mathbb{N}$ and suppose that any poset of cardinality n has a maximal element. Let X be a poset of cardinality $n+1$ and choose an arbitrary element $x \in X$. Let $Y = X \setminus \{x\}$, then Y is a poset of cardinality n so by the induction hypothesis has a maximal element m_Y, and clearly $m_Y \neq x$.

   We have two possibilities now:

   - If $m_Y \leqslant x$, then x is a maximal element of X. Why? Suppose that x is not maximal in X, so that there exists $z \in X$ such that $z \neq x$ and $x \leqslant z$. Since $z \neq x$, we must have $z \in Y$. If $z = m_Y$, then $z \leqslant x$ and $x \leqslant z$ so $z = x$, contradiction. So $z \neq m_Y$, and $m_Y \leqslant x$ and $x \leqslant z$, so $m_Y \leqslant z$, contradicting the maximality of m_Y in Y.
   - Otherwise, (if it is not true that $m_Y \leqslant x$), m_Y is a maximal element of X. Why? Suppose there exists $z \in X$ such that $z \neq m_Y$ and $m_Y \leqslant z$. Since $m_Y \leqslant x$ is not true, we have $z \neq x$, so $z \in Y$, contradicting the maximality of m_Y in Y.

   In either case we found a maximal element for X. □

   Solution. An alternative approach is to proceed by contradiction: suppose $(X, \leqslant)$ is a nonempty finite poset that does not have a maximal element. Use this to construct an unbounded chain of elements of X, contradicting finiteness. □

5. Let $n \in \mathbb{N}$, $X = \mathbb{R}^n$ with the dot product $\cdot$, $\|x\| = \sqrt{x \cdot x}$ for $x \in X$, and $d(x, y) = \|x - y\|$ for $x, y \in X$. Then (X, d) is a metric space. (The function d is called the *Euclidean metric* or *ℓ^2 metric* on $\mathbb{R}^n$.)

   [*Hint*: The Cauchy–Schwarz inequality can be useful for checking the triangle inequality.]

   Solution. We have

   (a) $d(x, y) = \|x - y\| = \sqrt{(x - y) \cdot (x - y)} = \sqrt{(-1)^2 (y - x) \cdot (y - x)} = \|y - x\| = d(y, x)$;

   (b) Let $u = x - t$ and $v = t - y$, then we are looking to show that $\|u + v\| \leqslant \|u\| + \|v\|$. But:

   $$\begin{aligned} \|u + v\|^2 = (u + v) \cdot (u + v) = \|u\|^2 + 2\, u \cdot v + \|v\|^2 &\leqslant \|u\|^2 + 2\, |u \cdot v| + \|v\|^2 \\ &\leqslant \|u\|^2 + 2\, \|u\|\, \|v\| + \|v\|^2 = \big(\|u\| + \|v\|\big)^2, \end{aligned}$$

   where the last inequality sign comes from the Cauchy–Schwarz inequality.

(c) $d(x,y) = 0$ iff $(x-y)\cdot(x-y) = 0$ iff $x - y = 0$ iff $x = y$.

$\square$

6. Let X be a nonempty set and define

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that (X,d) is a metric space. (The function d is called the *discrete metric* on X.)

Solution. It is clear from the definition that $d(y,x) = d(x,y)$ and that $d(x,y) = 0$ iff $x = y$. For the triangle inequality, take $x, y, t \in X$ and consider the different cases:

$x = y$	$x = t$	$t = y$	$d(x,y)$	$d(x,t) + d(t,y)$
True	True	True	0	$0 + 0 = 0$
True	False	False	0	$1 + 1 = 2$
False	True	False	1	$1 + 0 = 1$
False	False	True	1	$0 + 1 = 1$
False	False	False	1	$1 + 1 = 2$

In all cases we see that $d(x,y) \leqslant d(x,t) + d(t,y)$. $\square$

7. Let (X,d) be a metric space and define

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

Prove that (X,d') is a metric space.

[*Hint*: Before tackling the triangle inequality, show that if $a, b, c \in \mathbb{R}_{\geqslant 0}$ satisfy $c \leqslant a + b$, then $\frac{c}{1+c} \leqslant \frac{a}{1+a} + \frac{b}{1+b}$.]

Solution. It is clear from the definition that $d'(x,y) = d'(y,x)$ and that $d'(x,y) = 0$ iff $d(x,y) = 0$ iff $x = y$.

For the triangle inequality, apply the inequality in the hint with $c = d(x,y)$, $a = d(x,t)$, $b = d(t,y)$. $\square$

8. Draw the unit open balls in the metric spaces $(\mathbb{R}^2, d_1)$ (Example 2.3), $(\mathbb{R}^2, d_2)$ (Exercise 2.14), and $(\mathbb{R}^2, d_\infty)$ (Example 2.4).

Solution. The Manhattan unit open ball is the interior of the square with vertices $(1,0)$, $(0,-1)$, $(-1,0)$, and $(0,1)$.

The Euclidean unit open ball is the interior of the unit circle centred at $(0,0)$.

The sup metric unit open ball is the interior of the square with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$. $\square$

9. Is the word "finite" necessary in the statement of Proposition 2.12? If no, give a proof of the statement without "finite". If yes, give an example of an infinite collection of open sets whose intersection is not an open set.

Solution. The word "finite" is necessary. For a counterexample to the more general statement, for each $n \in \mathbb{N}$ take $U_n = (-1/n, 1/n)$ as an open set in $\mathbb{R}$ with the Euclidean metric. I claim that

$$U := \bigcap_{n \in \mathbb{N}} U_n = \{0\}.$$

This can be proved by contradiction: suppose $u \in U$, $u \neq 0$. Let $n \in \mathbb{N}$ be such that $n \geqslant \frac{1}{|u|}$. Then $|u| \geqslant \frac{1}{n}$, therefore $u \notin (-1/n, 1/n) = U_n$, contradiction.

Finally, U is not open: for any $r \in \mathbb{R}_{>0}$, $\frac{r}{2} \in \mathbb{B}_r(0)$ but $\frac{r}{2} \notin \{0\} = U$, so $\mathbb{B}_r(0)$ is not a subset of U. $\square$