Tutorial Week 12

Topics: Convergence in the norm, maps of finite rank, spectrum

1. For $n \in \mathbb{N}$, consider the function $f_n\colon [0,2] \longrightarrow \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leqslant x \leqslant \frac{1}{n} \\ -n^2\left(x - \frac{2}{n}\right) & \text{if } \frac{1}{n} < x \leqslant \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} < x \leqslant 2. \end{cases}$$

   (You might want to graph f_1, f_2, f_3 to get a feel for what the functions look like.)

   Find the pointwise limit $f(x)$ of $\big(f_n(x)\big)$ for all $x \in [0,2]$.

   Show that (f_n) does not converge to f with respect to the L^1 norm.

2. Given a subset $S \subseteq [0,1]$, let $\mathbf{1}_S\colon [0,1] \longrightarrow \mathbb{R}$ denote the *characteristic function* of S, that is

$$\mathbf{1}_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

   Consider the sequence of functions (g_n) defined as follows: write $n \in \mathbb{N}$ in the form

$$n = 2^k + \ell, \qquad k, \ell \in \mathbb{Z}_{\geqslant 0}, 0 \leqslant \ell < 2^k,$$

   then define $g_n\colon [0,1] \longrightarrow \mathbb{R}$ by

$$g_n = \mathbf{1}_{[\ell/2^k,(\ell+1)/2^k]}.$$

   (You might want to graph $g_1, \ldots, g_5$ to get a feel for what the functions look like.)

   Show that (g_n) converges to the constant function zero with respect to the L^1 norm, but that $\big(g_n(x)\big)$ does not converge for any $x \in [0,1]$.

3. Let $R(H)$ denote the set of all maps $f \in B(H)$ of finite rank on a complex Hilbert space H.

   Prove that $R(H)$ is a vector subspace of $B(H)$.

4. Prove that if $f \in R(H)$ and $g_1, g_2 \in B(H)$ then $g_2 \circ f \circ g_1 \in R(H)$.

5. Prove that if $f \in R(H)$ then $f^* \in R(H)$.

   [*Hint*: Use Proposition 4.35.]

6. Recall the right shift operator $R\colon \ell^2 \longrightarrow \ell^2$

$$R(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots).$$

   (a) Prove that R has no complex eigenvalues.

   (b) Prove that $0 \in \sigma(R)$.

   (c) Is R a compact map?

7. Let H be a complex Hilbert space and let

$$\mathrm{GL}(H) = \{f \in B(H) : f \text{ is invertible}\}.$$

For $f \in \mathrm{GL}(H)$, prove that

$$\mathbb{B}_r(f) \subseteq \mathrm{GL}(H) \qquad \text{where } r = \left\|f^{-1}\right\|^{-1}.$$

[*Hint*: Given $g \in \mathbb{B}_r(f)$, consider $i := -f^{-1} \circ (g - f)$ and use Proposition 4.40 to show that $\mathrm{id}_H - i$ is invertible.]

Conclude that $\mathrm{GL}(H)$ is an open subset of $B(H)$.

8. Prove that the spectrum of any $f \in B(H)$ is a compact set.

[*Hint*: Use Question 7 to show that the resolvent $\rho(f)$ is an open subset of $\mathbb{C}$, then use Corollary 4.41.]

9. Let V, W be normed spaces, with V Banach, and let $f \in B(V, W)$. Suppose that there exists a constant $c > 0$ such that

$$\|f(v)\|_W \geqslant c\,\|v\|_V \qquad \text{for all } v \in V.$$

Then $\mathrm{im}(f)$ is a closed subspace of W.

10. Let $f \in B(H)$ be a self-adjoint map on a complex Hilbert space H and let $a + ib \in \mathbb{C}$. Prove that

$$\left\|\big(f - (a + ib)\,\mathrm{id}_H\big)(x)\right\| \geqslant |b|\,\|x\| \qquad \text{for all } x \in H.$$

[*Hint*: Expand $\|(f - (a + ib)\,\mathrm{id}_H)(x)\|^2$ using the inner product, take advantage of $f^* = f$, and manipulate until you get a sum of two squares, one of which is $b^2\|x\|^2$.]