Tutorial Week 11

Topics: Self-adjoint maps, uniform norm, pointwise and uniform convergence.

1. Let $f \in B(H, H)$ with H a Hilbert space. Then the maps

$$p = f^* \circ f \qquad \text{and} \qquad s = f + f^*$$

are self-adjoint.

2. The composition of two self-adjoint maps f, g on a Hilbert space is self-adjoint if and only if the maps commute.

3. Let $f \in B(H, H)$ with H a Hilbert space. Suppose that f is invertible with continuous inverse. Then the adjoint f^* is invertible and

$$\left(f^*\right)^{-1} = \left(f^{-1}\right)^*.$$

4. Let B be an orthonormal system in a Hilbert space H. Prove that B is an orthonormal basis if and only if:

for every $x \in H$, if $\langle x, y \rangle = 0$ for all $y \in B$, then $x = 0$.

5. For each $n \in \mathbb{N}$ define $f_n \colon [0, 1] \longrightarrow \mathbb{R}$ by

$$f_n(x) = \frac{x^2}{1 + nx}.$$

Convince yourself that each f_n is continuous.

Find the pointwise limit f of the sequence (f_n) and determine whether the sequence converges uniformly to f.

6. For each $n \in \mathbb{N}$ define $f_n \colon [0, 1] \longrightarrow \mathbb{R}$ by

$$f_n(x) = \frac{1 - x^n}{1 + x^n}.$$

Convince yourself that each f_n is continuous.

Find the pointwise limit f of the sequence (f_n) and determine whether the sequence converges uniformly to f.

7. Suppose that the Weierstraß Approximation Theorem holds for $K = [0, 1]$.

Prove that the Theorem holds for any closed interval $[a, b]$ with $a < b$.

[*Hint*: Find a polynomial function of degree one $\varphi \colon [0, 1] \longrightarrow [a, b]$ that is surjective and use it and its inverse to move between functions on $[0, 1]$ and functions on $[a, b]$.]

See the back of this page for a very optional Question 8.

8. (*) Prove that for any $x \in \mathbb{R}$ and for any $n \in \mathbb{Z}_{\geqslant 0}$ we have

(a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1;$

(b) $\displaystyle\sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k} = nx;$

(c) $\displaystyle\sum_{k=0}^{n} k^2 \binom{n}{k} x^k (1-x)^{n-k} = n(n-1)x^2 + nx;$

(d) $\displaystyle\delta^2 \sum_{k\,:\,|k/n-x|\geqslant\delta} \binom{n}{k} x^k (1-x)^{n-k} \leqslant \frac{1}{4n}$ for all $\delta > 0$.

[*Hint*: For (b), note that $k\binom{n}{k} = n\binom{n-1}{k-1}$.
For (c), start by showing that $\displaystyle\sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} = n(n-1)x^2$.
For (d), use the fact that $\delta^2 \leqslant \left(x - k/n\right)^2$ for all k such that $|k/n - x| \geqslant \delta$, so that the sum in question is bounded above by $\displaystyle\sum_{k=0}^{n} \left(x-k/n\right)^2 \binom{n}{k} x^k (1-x)^{n-k}$.]