# Tutorial Week 10

**Topics:** Projections; adjoint maps

1. Let $V$ be a normed space and $\varphi, \psi$ be commuting projections: $\varphi \circ \psi = \psi \circ \varphi$. Prove that $\varphi \circ \psi$ is a projection with image $\operatorname{im}\varphi \cap \operatorname{im}\psi$.

2. Let $\varphi$ be a nonzero orthogonal projection (that is, $\varphi$ is not the constant function 0) on an inner product space $V$. Prove that $\|\varphi\| = 1$.

3. Let $S$ be a subset of a Hilbert space $H$. Prove that $\operatorname{Span}(S)$ is dense in $H$ if and only if $S^{\perp} = 0$.

4. Let $V, W$ be inner product spaces and let $f \in B(V, W)$. Prove that

$$\|f\| = \sup_{\|v\|_V = \|w\|_W = 1} \left|\langle f(v), w\rangle_W\right|.$$

[*Hint*: Use Exercise 4.2 which says that $\|v\| = \sup_{\|w\|=1} \left|\langle v, w\rangle\right|$.]

5. Recall that the adjoint $f^*\colon Y \longrightarrow X$ of a continuous linear map $f\colon X \longrightarrow Y$ of Hilbert spaces satisfies the property

$$\big\langle f(x), y\big\rangle_Y = \big\langle x, f^*(y)\big\rangle_X \qquad \text{for all } x \in X, y \in Y.$$

Prove that for all $\alpha \in \mathbb{F}$ we have

$$\big(\alpha f\big)^* = \overline{\alpha} f^*.$$

6. Given continuous linear maps $g\colon X \longrightarrow Y$ and $f\colon Y \longrightarrow Z$ of Hilbert spaces, prove that

$$\big(f \circ g\big)^* = g^* \circ f^*.$$

7. Prove that for any Hilbert space $X$ we have

$$\operatorname{id}_X^* = \operatorname{id}_X.$$

8. Prove that for any continuous linear map $f\colon X \longrightarrow Y$ of Hilbert spaces, we have

$$\big(f^*\big)^* = f.$$

9. Let $f\colon X \longrightarrow Y$ be a continuous linear map of Hilbert spaces. Prove that

$$\ker\big(f^*\big) = \big(\operatorname{im} f\big)^{\perp} \qquad \text{and} \qquad \overline{\operatorname{im}\big(f^*\big)} = \big(\ker f\big)^{\perp}.$$

10. Consider the function $g\colon \ell^2 \longrightarrow \mathbb{F}$ given by

$$g(x) = \sum_{n=1}^{\infty} \frac{x_n}{n^2}.$$

(a) Find $y \in \ell^2$ such that

$$g(x) = \langle x, y\rangle \qquad \text{for all } x \in \ell^2.$$

(b) Deduce that $g$ is linear and bounded and find its norm $\|g\|$.

[*Hint*: You may use without proof the fact that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.]