Tutorial Week 09

Topics: more sequence spaces; inner product spaces.

1. Consider the map $\pi_1 \colon \mathbb{F}^{\mathbb{N}} \longrightarrow \mathbb{F}$ given by
$$\pi_1\big((a_n)\big) = a_1.$$
   (a) Show that π_1 is linear.
   (b) Prove that the restriction of π_1 to ℓ^∞ or to ℓ^p for $p \geqslant 1$ is continuous and surjective.

2. Consider the left shift map $L \colon \mathbb{F}^{\mathbb{N}} \longrightarrow \mathbb{F}^{\mathbb{N}}$ given by $L\big((a_n)\big) = (a_{n+1})$, that is
$$L(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots).$$
   (a) Prove that L is a surjective linear map. What is the kernel of L?
   (b) Prove that for all $p \geqslant 1$ and for $p = \infty$, the restriction of L to ℓ^p is a surjective continuous map onto ℓ^p.
   (c) Define the right shift map $R \colon \mathbb{F}^{\mathbb{N}} \longrightarrow \mathbb{F}^{\mathbb{N}}$ and prove that it is an injective linear map, the restriction of which is distance-preserving for any ℓ^p with $p \geqslant 1$ and $p = \infty$.
   (d) Check that $L \circ R = \mathrm{id}_{\mathbb{F}^{\mathbb{N}}} \neq R \circ L$.

3. (*) Consider the subset c of $\mathbb{F}^{\mathbb{N}}$ consisting of all convergent sequences (with any limit).
   (a) Convince yourself that c is a vector subspace of ℓ^∞.
   (b) Prove that $\lim \colon c \longrightarrow \mathbb{F}$ given by
$$(a_n) \longmapsto \lim_{n \longrightarrow \infty} (a_n)$$
   is a continuous surjective linear map.
   (c) Prove that the formula
$$J\big((a_n)\big) = R\big((a_n)\big) - \Big(\lim_{n \longrightarrow \infty} a_n\Big)(1, 1, \dots)$$
   defines a linear homeomorphism $J \colon c \longrightarrow c_0$. (Here R denotes the right shift map.)
   (d) Show that c is separable and find a Schauder basis for c.

4. For any $n \in \mathbb{N}$, give a linear distance-preserving map $\mathbb{F}^n \longrightarrow \ell^2$. (Take the Euclidean norm on $\mathbb{F}^n$.)

5. Let $\big(V, \langle \cdot, \cdot \rangle\big)$ be an inner product space. Prove that the inner product is a continuous function.

6. Let $\big(V, \langle \cdot, \cdot \rangle\big)$ be an inner product space. For any $v \in V$ we have
$$\|v\| = \sup_{\|w\|=1} \big|\langle v, w \rangle\big|.$$
   The supremum is in fact achieved by a well-chosen w.

7. Let $\big(V, \langle \cdot, \cdot \rangle\big)$ be an inner product space and let R, S be subsets of V.
   (a) Prove that $S \cap S^\perp = 0$.
   (b) Prove that if $R \subseteq S$ then $S^\perp \subseteq R^\perp$.
   (c) Prove that $S \subseteq \big(S^\perp\big)^\perp$.
   (d) Prove that $S^\perp = \overline{\mathrm{Span}(S)}^\perp$.

8. Let $\big(X, d\big)$ be a metric space and let $S \subseteq X$. Prove that $d_S(x) = 0$ if and only if $x \in \overline{S}$.