Tutorial Week 08

Topics: series in normed spaces; sequence spaces.

1. If a series in a normed space $\big(V, \|\cdot\|\big)$
$$\sum_{n=1}^{\infty} a_n$$
converges, and converges absolutely, then
$$\left\|\sum_{n=1}^{\infty} a_n\right\| \leqslant \sum_{n=1}^{\infty} \|a_n\|.$$

2. Give an example of a series that converges but does not converge absolutely.

3. If $f \in B(V, W)$ with V, W normed spaces, and the series
$$\sum_{n=1}^{\infty} \alpha_n v_n, \qquad \alpha_n \in \mathbb{F}, v_n \in V,$$
converges in V, then the series
$$\sum_{n=1}^{\infty} \alpha_n f(v_n)$$
converges in W to the limit
$$f\left(\sum_{n=1}^{\infty} \alpha_n v_n\right).$$

4. Prove that if $u = \big(u_n\big) \in \ell^\infty$ and $v = \big(v_n\big) \in \ell^1$, then
$$\sum_{n=1}^{\infty} |u_n v_n| \leqslant \|u\|_{\ell^\infty} \|v\|_{\ell^1}.$$

5. Consider the subset $c_0 \subseteq \mathbb{F}^{\mathbb{N}}$ of all sequences with limit 0:
$$c_0 = \big\{(a_n) \in \mathbb{F}^{\mathbb{N}} : (a_n) \longrightarrow 0\big\}.$$
Prove that c_0 is a closed subspace of ℓ^∞.

   Conclude that c_0 is a Banach space.

6. Prove that the space c_0 of sequences with limit 0 is separable, by finding a Schauder basis for c_0.

   [*Hint*: You needn't look too hard.]

7. Consider the space ℓ^∞ of bounded sequences.
   (a) Let $S \subseteq \ell^\infty$ be the subset of sequences (a_n) such that $a_n \in \{0, 1\}$ for all $n \in \mathbb{N}$. Prove that S is an uncountable set.

   [*Hint*: Mimic Cantor's diagonal argument.]

   (b) Use S to construct an uncountable set T of disjoint open balls in ℓ^∞.
   (c) Conclude that ℓ^∞ is not separable.