Tutorial Week 07

Topics: metric properties of normed spaces

1. Let $(V, \|\cdot\|)$ be a normed space and let $S \subseteq V$ be a subset. Then $\overline{\mathrm{Span}(S)}$ is the smallest closed subspace of V that contains S.

2. Let $(V, \|\cdot\|)$ be a normed space and take $r, s > 0$, $u, v \in V$, $\alpha \in \mathbb{F}^{\times}$. Show that
   (a) $\mathbb{B}_r(u+v) = \mathbb{B}_r(u) + \{v\}$;
   (b) $\alpha\,\mathbb{B}_1(0) = \mathbb{B}_{|\alpha|}(0)$;
   (c) $\mathbb{B}_r(v) = r\mathbb{B}_1(0) + \{v\}$;
   (d) $r\mathbb{B}_1(0) + s\mathbb{B}_1(0) = (r+s)\mathbb{B}_1(0)$;
   (e) $\mathbb{B}_r(u) + \mathbb{B}_s(v) = \mathbb{B}_{r+s}(u+v)$;
   (f) $\mathbb{B}_1(0)$ is a convex subset of V;
   (g) any open ball in V is convex.

3. Let $(V, \|\cdot\|)$ be a normed space and let S, T be subsets of V and $\alpha \in \mathbb{F}$. Prove that
   (a) If S and T are bounded, so are $S+T$ and $\alpha\,S$.
   (b) If S and T are totally bounded, so are $S+T$ and $\alpha\,S$.
   (c) If S and T are compact, so are $S+T$ and $\alpha\,S$.

4. Let $f \in B(V, W)$, that is a continuous linear transformation $f\colon V \longrightarrow W$.
   (a) If U is a subspace of V, then its image $f(U)$ is a subspace of W.
   (b) If U is a closed subspace of W, then its preimage $f^{-1}(U)$ is a closed subspace of V.
   (c) If S is a convex subset of V, then its image $f(S)$ is a convex subset of W.
   (d) If S is a convex subset of W, then its preimage $f^{-1}(S)$ is a convex subset of V.

5. Prove that the following subset is a closed subspace of ℓ^1:
$$S = \left\{ (a_n) \in \ell^1 : \sum_{n=1}^{\infty} a_n = 0 \right\}.$$

6. Suppose $1 \leqslant p \leqslant q$. Prove that
$$\ell^p \subseteq \ell^q.$$
   Show that if $p < q$ then the inclusion is strict: $\ell^p \subsetneq \ell^q$.