Tutorial Week 06

Topics: compact sets, normed spaces, inequalities galore.

1. Let (X, d_X) and (Y, d_Y) be metric spaces, and let d be any conserving metric on $X \times Y$.
   (a) Prove that if X and Y are compact, then $X \times Y$ is compact.
   [*Hint*: If you're not sure where to start, try sequential compactness.]
   (b) Does the converse hold?

2. Let C be a nonempty compact subset of a metric space (X, d). Prove that there exist points $a, b \in C$ such that
$$d(a, b) = \sup\{d(x, y) : x, y \in C\}.$$
   In other words, the diameter of C is realised as the distance between two points of C.

3. A subset S of a vector space V over $\mathbb{F}$ is said to be *convex* if for all $v, w \in S$ and all $a, b \in \mathbb{R}_{\geqslant 0}$ such that $a + b = 1$, we have
$$av + bw \in S.$$
   (In other words, for any two points in S, the line segment joining the two points is entirely contained in S.)
   Show that:
   (a) Any subspace W of V is convex.
   (b) The intersection of an arbitrary collection of convex sets is convex.
   (c) Any interval $I \subseteq \mathbb{R}$ is convex.

4. If V is a vector space over $\mathbb{F}$ and $S \subseteq V$ is a convex set, we say that a function $f: S \longrightarrow \mathbb{R}$ is *convex* if for all $v, w \in S$ and all $a, b \in \mathbb{R}_{\geqslant 0}$ such that $a + b = 1$, we have
$$f(av + bw) \leqslant af(v) + bf(w).$$
   Prove that, if $(V, \|\cdot\|)$ is a normed space, then $f: V \longrightarrow \mathbb{R}$ given by $f(v) = \|v\|$ is a convex function.

5. (a) Prove that the functions
$$\begin{array}{lll} \text{(i)} & f: (0,\infty) \longrightarrow \mathbb{R}, & f(x) = x^p, \quad p \geqslant 1 \text{ fixed}, \\ \text{(ii)} & \exp: \mathbb{R} \longrightarrow \mathbb{R}, & \exp(x) = e^x, \end{array}$$
   are convex.
   [*Hint*: Use the second-derivative criterion from Q7.]
   (b) Conclude that for any $p \geqslant 1$, any $x, y \geqslant 0$ and any $a, b \geqslant 0$ such that $a + b = 1$, we have
$$(ax + by)^p \leqslant ax^p + by^p.$$
   (c) Conclude that for any $x, y \geqslant 0$ and any $a, b \geqslant 0$ such that $a + b = 1$, we have
$$x^a y^b \leqslant ax + by.$$
   [*Hint*: Set $x = e^s$, $y = e^t$.]
   (d) Show that for any $p \geqslant 1$ and any $x, y \geqslant 0$, we have
$$x^p + y^p \leqslant (x + y)^p.$$
   [*Hint*: Let $t = x/y$ and compare derivatives to show that $t^p + 1 \leqslant (t + 1)^p$.]

6. (*) Let $p \geqslant 1$, $q > 0$, $x, y \geqslant 0$, and $a, b \geqslant 0$ such that $a + b = 1$.

   Prove that

$$\begin{aligned}
\min\{x, y\} &\leqslant \left(ax^{-q} + by^{-q}\right)^{-1/q} \\
&\leqslant x^a y^b \\
&\leqslant \left(ax^{1/p} + by^{1/p}\right)^p \\
&\leqslant ax + by \\
&\leqslant \left(ax^p + by^p\right)^{1/p} \\
&\leqslant \max\{x, y\}.
\end{aligned}$$

7. (*) Let $I \subseteq \mathbb{R}$ be an interval and let $f\colon I \longrightarrow \mathbb{R}$ be a twice-differentiable function.

   The aim of this Exercise is to check the familiar calculus fact: f is convex if and only if $f''(x) \geqslant 0$ for all $x \in I$.

   It was heavily inspired by Alexander Nagel's Wisconsin notes:

   https://people.math.wisc.edu/~ajnagel/convexity.pdf

   (a) For any $s, t \in I$ with $s < t$, define the linear function $L_{s,t}\colon [s, t] \longrightarrow \mathbb{R}$ by

$$L_{s,t}(x) = f(s) + \left(\frac{x - s}{t - s}\right)\left(f(t) - f(s)\right).$$

   Convince yourself that this is the equation of the secant line joining $\left(s, f(s)\right)$ to $\left(t, f(t)\right)$.

   Prove that f is convex on I if any only if

$$f(x) \leqslant L_{s,t}(x) \qquad \text{for all } s, t \in I \text{ such that } s < t \text{ and all } s \leqslant x \leqslant t.$$

   (b) Check that for all $s, t \in I$ such that $s < t$ we have

$$L_{s,t}(x) - f(x) = \frac{x - s}{t - s}\left(f(t) - f(x)\right) - \frac{t - x}{t - s}\left(f(x) - f(s)\right).$$

   (c) Use the Mean Value Theorem for f twice to prove that there exist ξ, ζ with $x < \xi < t$ and $s < \zeta < x$ such that

$$L_{s,t}(x) - f(x) = \frac{(t - x)(x - s)}{t - s}\left(f'(\xi) - f'(\zeta)\right).$$

   (d) Use the Mean Value Theorem once more to conclude that if $f''(x) \geqslant 0$ for all $x \in I$, then f is convex on I.

   (e) Now we prove the converse. From this point on, assume that $f\colon I \longrightarrow \mathbb{R}$ is twice-differentiable and convex, and let $s, t \in I^{\circ}$.

      1. Show that if $s < x < t$ then

$$\frac{f(x) - f(s)}{x - s} \leqslant \frac{f(t) - f(x)}{t - x}.$$

      2. Conclude that if $s < x_1 < x_2 < t$ then

$$\frac{f(x_1) - f(s)}{x_1 - s} \leqslant \frac{f(t) - f(x_2)}{t - x_2}.$$

      3. Conclude that if $s < t$ then $f'(s) \leqslant f'(t)$, and finally that $f''(x) \geqslant 0$ on I.