Tutorial Week 05

Topics: connected, bounded, compact sets.
Note: You can skip the starred questions on a first pass and come back to them later.

1. Let A and C be connected subsets of a metric space (X,d). Prove that if $A \cap C \neq \varnothing$, then $A \cup C$ is connected.

2. Let (X,d) be a metric space. Suppose $A \subseteq X$ is a connected subset and $\{C_i : i \in I\}$ is an arbitrary collection of connected subsets of X such that $A \cap C_i \neq \varnothing$ for all $i \in I$. Then

$$A \cup \bigcup_{i \in I} C_i$$

is a connected subset of X.

[*Hint*: Use the argument from Q1.]

3. (*) Let (X,d) be a metric space. Suppose $\{C_n : n \in \mathbb{N}\}$ is a countable collection of connected subsets of X such that $C_n \cap C_{n+1} \neq \varnothing$ for all $n \in \mathbb{N}$. Then

$$\bigcup_{n \in \mathbb{N}} C_n$$

is a connected subset of X.

[*Hint*: Build the union inductively, and use Q1 and Q2.]

4. (*) Let (X,d) be a metric space and define $x \sim x'$ if there exists a connected subset $C \subset X$ such that $x, x' \in C$.

Prove that this is an equivalence relation on the set X, thereby partitioning X into a disjoint union of maximal connected subsets (these are called the *connected components* of X).

[*Hint*: Recall that an equivalence relation has three defining axioms: (a) $x \sim x$ for all $x \in X$; (b) if $x \sim x'$ then $x' \sim x$; (c) if $x \sim x'$ and $x' \sim x''$ then $x \sim x''$.]

5. Give explicit continuous surjective functions $f : \mathbb{R} \longrightarrow I$, where I is:

(a) $\mathbb{R}$ (b) $(0,\infty)$ (c) $(-\infty,0)$ (d) $(-\infty,0]$ (e) $[-1,1]$
(f) $(0,1]$ (g) $[0,1)$ (h) $(-\pi/2,\pi/2)$ (i) $\{0\}$.

[*Hint*: Draw some functions you know from calculus and see what their ranges are.]

6. (*) Let (X,d) be a metric space.

If A and B are bounded sets with $A \cap B \neq \varnothing$, then

$$\operatorname{diam}(A \cup B) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B).$$

7. Let C be a closed subset of a compact subset K of a metric space (X,d). Prove that C is compact.

[*Hint*: $K \subseteq X = C \cup \big(X \setminus C\big)$.]

8. Let K and L be compact subsets of a metric space (X,d). Prove that $K \cup L$ is compact.