Tutorial Week 04

Topics: completeness, uniform continuity.

1. Let (X, d_X) and (Y, d_Y) be metric spaces and let d be a conserving metric on $X \times Y$.
   (a) Prove that the sequence $\big((x_n, y_n)\big)$ is Cauchy in $X \times Y$ if and only if (x_n) is Cauchy in X and (y_n) is Cauchy in Y.
   (b) Prove that if X and Y are complete then $X \times Y$ is complete. Is the converse true?

2. Any distance-preserving function is uniformly continuous.

3. Check (directly from the definition of uniform continuity) that $f\colon \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ given by $f(x) = \frac{1}{x}$ is not uniformly continuous.

4. Let $f\colon X \longrightarrow Y$ be a uniformly continuous function between two metric spaces and suppose $(x_n) \sim (x'_n)$ are equivalent sequences in X. Prove that $\big(f(x_n)\big) \sim \big(f(x'_n)\big)$ as sequences in Y.

   Does the conclusion hold if f is only assumed to be continuous?

5. Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \longrightarrow Y$ a surjective continuous function. Suppose that X is complete and for all $x_1, x_2 \in X$ we have
$$d_X\big(x_1, x_2\big) \leqslant d_Y\big(f(x_1), f(x_2)\big).$$
   (a) Prove that Y is complete.

   In particular, distance-preserving maps preserve completeness.

   (b) Do you feel strongly that uniformly continuous functions ought to preserve completeness? (After all, they preserve Cauchy sequences, and completeness is defined in terms of Cauchy sequences.)

   Prove that $f\colon \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$ given by $f(x) = \arctan(x)$ is uniformly continuous, but...

6. Any Cauchy sequence (x_n) is *bounded*, that is there exists $C \geqslant 0$ such that $d(x_n, x_m) \leqslant C$ for all $n, m \in \mathbb{N}$.

7. Suppose A and B are abelian groups. A function $f\colon A \longrightarrow B$ is called *additive* if $f(a + b) = f(a) + f(b)$.
   (a) Prove that every additive function $f\colon A \longrightarrow B$ satisfies
$$f(0) = 0 \qquad \text{and} \qquad f(-a) = -f(a).$$
   (b) Let V be a $\mathbb{Q}$-vector space. Prove that every additive function $f\colon \mathbb{Q} \longrightarrow V$ is $\mathbb{Q}$-linear.
   (c) What can you say (and prove) about **continuous** additive functions $\mathbb{R} \longrightarrow \mathbb{R}$?
   (d) Suppose that $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is additive and continuous at 0. Prove that f is continuous on $\mathbb{R}$, and conclude that f is $\mathbb{R}$-linear.
   (e) Let B be a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. (Recall from Exercise 1.5 that B is uncountable.) Use two distinct irrational elements of B to construct a $\mathbb{Q}$-linear transformation $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ that is not $\mathbb{R}$-linear.

   If you would (and why wouldn't you?), follow the rabbit:

   https://en.wikipedia.org/wiki/Cauchy%27s_functional_equation