Tutorial Week 03

Topics: convergence of sequences, continuous functions, nowhere dense sets, equivalence of metrics

1. Let X and Y be two metric spaces and endow the Cartesian product $X \times Y$ with the Manhattan metric from Example 2.3. Prove that a sequence $\big((x_n, y_n)\big)$ in $X \times Y$ converges to (x, y) if and only if (x_n) converges to x and (y_n) converges to y.

2. Let (x_n) be a sequence in X, let $\varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ be an injective function, and consider the sequence $(y_n) = \big(x_{\varphi(n)}\big)$ in X. Prove that if (x_n) converges to x, then so does (y_n).

   Does the converse hold?

3.
   (a) Let $f\colon X \longrightarrow Y$ be a function between two sets X and Y, and let $S \subseteq Y$. Prove that

   $$f^{-1}(S) = X \setminus f^{-1}(Y \setminus S).$$

   (b) Let $f\colon X \longrightarrow Y$ be a function between metric spaces. Prove that f is continuous if and only if: for any closed subset $C \subseteq Y$, the inverse image $f^{-1}(C) \subseteq X$ is a closed subset.

4. Show that if $f\colon X \longrightarrow Y$ is a continuous map between metric spaces and $A \subseteq X$ then $f\big(\overline{A}\big) \subseteq \overline{f(A)}$.

5. Give $\mathbb{N}$ the metric induced from $\mathbb{R}$. Let (X, d) be a metric space and (x_n) a sequence in X. Prove that (x_n) is a continuous function $\mathbb{N} \longrightarrow X$.

6.
   (a) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be functions, where X, Y, Z are sets, and let $S \subseteq Z$. Then

   $$f^{-1}\big(g^{-1}(S)\big) = (g \circ f)^{-1}(S).$$

   (b) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be continuous functions, where X, Y, Z are metric spaces. Prove that $g \circ f\colon X \longrightarrow Z$ is continuous.

7. Let $f\colon X \longrightarrow Y$ be a continuous map between metric spaces and let $S \subseteq Y$ be such that $f(X) \subseteq S$. Endowing S with the metric induced from Y, show that $f\colon X \longrightarrow S$ is continuous.

8. Let $g_1\colon X \longrightarrow Y_1$ and $g_2\colon X \longrightarrow Y_2$ be continuous maps, with X, Y_1, Y_2 metric spaces.

   Define $f\colon X \longrightarrow Y_1 \times Y_2$ by $f(x) = \big(g_1(x), g_2(x)\big)$. Endow $Y_1 \times Y_2$ with the Manhattan metric.

   Show that f is continuous if and only if both g_1 and g_2 are continuous.

9. If A and B are subsets of a metric space (X, d), then

   $$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

10. Let (X, d) be a metric space.

    (a) Prove that any subset of a nowhere dense subset of X is nowhere dense in X.

    (b) Prove that a subset $N \subseteq X$ is nowhere dense if and only if $X \setminus \overline{N}$ is dense in X.

(c) Prove that the union of any finite collection of nowhere dense subsets of X is nowhere dense in X.

11. Let X be a set.

  (a) Show that the relation "d_1 is finer than d_2" on metrics on X gives rise to a relation "$[d_1]$ is finer than $[d_2]$" on equivalence classes of metrics on X.

  (b) Show that the latter is a partial order on the set of equivalence classes of metrics on X.

  (c) In the statement from part (b), can we remove the words "equivalence classes of"?

  (d) Show that the partial order from part (b) has a unique maximal element.