Tutorial Week 02

Topics: infinite-dimensional vector spaces, metrics, open sets.

1. Let $\mathbb{R}^\infty$ be the set of arbitrary sequences $(x_1, x_2, \dots)$ of elements of $\mathbb{R}$.

   This is a vector space under the naturally-defined addition of sequences and multiplication by a scalar.

   Let $e_j \in \mathbb{R}^\infty$ be the sequence whose j-th entry is 1, and all the others are 0. Describe the subspace $\operatorname{Span}\{e_1, e_2, \dots\}$ of $\mathbb{R}^\infty$. Is the set $\{e_1, e_2, \dots\}$ a basis of $\mathbb{R}^\infty$?

2. Let $V = \mathbb{R}$ viewed as a vector space over $\mathbb{Q}$.

   Let $\alpha \in \mathbb{R}$. Show that the set $T = \{\alpha^n : n \in \mathbb{N}\}$ is $\mathbb{Q}$-linearly independent if and only if α is transcendental.

   (Note: An element $\alpha \in \mathbb{R}$ is called algebraic if there exists a monic polynomial $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$. An element $\alpha \in \mathbb{R}$ is called transcendental if it is not algebraic.)

3. Let W be a $\mathbb{Q}$-vector space with a countable basis B. Show that W is a countable set.

   [*Hint*: Use Exercise 1.2.]

   Conclude that $\mathbb{R}$ does not have a countable basis as a vector space over $\mathbb{Q}$.

4. Let $(X, \leqslant)$ be a nonempty finite poset. (This just means that X is a nonempty finite set with a partial order $\leqslant$.) Prove that X has a maximal element.

   [*Hint*: You could, for instance, use induction on the number of elements of X.]

5. Let $n \in \mathbb{N}$, $X = \mathbb{R}^n$ with the dot product $\cdot$, $\|x\| = \sqrt{x \cdot x}$ for $x \in X$, and $d(x, y) = \|x - y\|$ for $x, y \in X$. Then (X, d) is a metric space. (The function d is called the *Euclidean metric* or ℓ^2 *metric* on $\mathbb{R}^n$.)

   [*Hint*: The Cauchy–Schwarz inequality can be useful for checking the triangle inequality.]

6. Let X be a nonempty set and define

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that (X, d) is a metric space. (The function d is called the *discrete metric* on X.)

7. Let (X, d) be a metric space and define

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

   Prove that (X, d') is a metric space.

   [*Hint*: Before tackling the triangle inequality, show that if $a, b, c \in \mathbb{R}_{\geqslant 0}$ satisfy $c \leqslant a + b$, then $\frac{c}{1+c} \leqslant \frac{a}{1+a} + \frac{b}{1+b}$.]

8. Draw the unit open balls in the metric spaces $(\mathbb{R}^2, d_1)$ (Example 2.3), $(\mathbb{R}^2, d_2)$ (Exercise 2.14), and $(\mathbb{R}^2, d_\infty)$ (Example 2.4).

9. Is the word "finite" necessary in the statement of Proposition 2.12? If no, give a proof of the statement without "finite". If yes, give an example of an infinite collection of open sets whose intersection is not an open set.