Assignment 2

Note: Due Friday 13 October at 20:00 on Canvas & Gradescope. Please read the instructions given on Canvas. The questions have varying lengths and do not all count for the same number of marks; you may assume that longer questions are worth more.

1. Let U, V, W be normed spaces over $\mathbb{F}$.

   Suppose $\beta\colon U \times V \longrightarrow W$ is a continuous bilinear map.

   Consider the linear function $\beta_U\colon U \longrightarrow \operatorname{Hom}(V, W)$ given by $\beta_U(u) = f_u$, where

   $$f_u\colon V \longrightarrow W \qquad \text{is defined by } f_u(v) = \beta(u, v).$$

   (a) Prove that for any $u \in U$, $f_u \in B(V, W)$, in other words f_u is continuous.

   (b) By part (a) we can think of β_U as a function $U \longrightarrow B(V, W)$.
   Prove that $\beta_U\colon U \longrightarrow B(V, W)$ is continuous.

Solution. Before we start, we establish a useful statement: if a bilinear map β is continuous then β is a *bounded bilinear map*, that is there exists $c > 0$ such that

$$\|\beta(u, v)\|_W \leqslant c\,\|u\|_U\,\|v\|_V \qquad \text{for all } u \in U, v \in V.$$

(More can be said, see Exercise 3.23, but also Exercise 3.24 to dispel any hope at uniform continuity in this setting.)

To prove the statement, suppose β is continuous but not bounded. Then for every $n \in \mathbb{N}$ there exist vectors $u_n \in U$ and $v_n \in V$ such that

$$\|\beta(u_n, v_n)\|_W > n^2\,\|u_n\|_U\,\|v_n\|_V.$$

This forces u_n, v_n to be nonzero. Let

$$u_n' = \frac{1}{n\,\|u_n\|_U}\,u_n \text{ and } v_n' = \frac{1}{n\,\|v_n\|_V}\,v_n.$$

We now prove $(u_n', v_n') \longrightarrow (0,0)$ but $\beta(u_n', v_n') \not\longrightarrow 0 = \beta(0,0)$ as $n \longrightarrow \infty$, which contradicts the continuity of β.

Since $\|u_n'\|_U = \|v_n'\|_V = 1/n$, it follows that

$$\|(u_n', v_n')\|_{U\times V} = \|u_n'\|_U + \|v_n'\|_V = \frac{1}{2n}.$$

Therefore, $\|(u_n', v_n')\| \longrightarrow 0$ and thus $(u_n', v_n') \longrightarrow (0,0)$ as $n \longrightarrow \infty$.

On the other hand, we have

$$\|\beta(u_n', v_n')\|_W = \left\|\beta\left(\frac{1}{n\,\|u_n\|_U}\,u_n, \frac{1}{n\,\|v_n\|_V}\,v_n\right)\right\|_W = \frac{\|\beta(u_n, v_n)\|_W}{n^2\,\|u_n\|_U\,\|v_n\|_V} > 1.$$

Hence $\beta(u_n', v_n') \not\longrightarrow 0$ as $n \longrightarrow \infty$.

Now we can address the two parts of the question.

(a) **First approach (direct):** Let $v \in V$. We prove that $f_u \colon V \longrightarrow W$ is continuous at v. (Note that, crucially, u remains fixed.)

Let $\varepsilon > 0$; as β is continuous at (u, v), there exists $\delta > 0$ such that

$$\text{if } \|(u, v_1) - (u, v)\|_{U \times V} < \delta, \text{ then } \|\beta(u, v_1) - \beta(u, v)\|_W < \varepsilon.$$

Therefore, if $\|v_1 - v\|_V < \delta$, then

$$\|(u, v_1) - (u, v)\|_{U \times V} = \|v_1 - v\|_V < \delta,$$

so that

$$\|f_u(v_1) - f_u(v)\|_W = \|\beta(u, v_1) - \beta(u, v)\|_W < \varepsilon.$$

Second approach (using boundedness): Let $\varepsilon > 0$; as β is continuous, it is bounded, so there exists $c > 0$ such that

$$\|\beta(u, v)\|_W \leqslant c\,\|u\|_U\,\|v\|_V \qquad \text{for all } u \in U, v \in V.$$

It follows that

$$\|f_u(v)\|_W = \|\beta(u, v)\|_W \leqslant c\,\|u\|_U\,\|v\|_V.$$

Since $c\|u\|_U$ is a constant independent of v, the linear transformation f_u is bounded and thus continuous.

(b) Let $\varepsilon > 0$; as β is continuous, it is bounded, so there exists $c > 0$ such that

$$\|\beta(u, v)\|_W \leqslant c\,\|u\|_U\,\|v\|_V \qquad \text{for all } u \in U, v \in V.$$

It follows that

$$\|\beta_U(u)\|_{B(V,W)} = \|f_u\|_{B(V,W)} = \sup_{\|v\|_V = 1} \|\beta(u, v)\|_W \leqslant c\,\|u\|_U.$$

Therefore, β_U is bounded and thus continuous. □

2. In Proposition 3.23 we saw that the function

$$\ell^1 \times \ell^\infty \longrightarrow \mathbb{F} \qquad \text{defined by } (u,v) \longmapsto \sum_{n=1}^{\infty} u_n v_n$$

is a continuous bilinear map.

(a) Show that there is a continuous linear function $\ell^1 \longrightarrow (c_0)^\vee$ that is an isometry. (Recall that $c_0 \subseteq \ell^\infty$ consists of all convergent sequences with limit 0.)

[*Hint*: It may be useful to prove surjectivity first, and then the distance-preserving property.]

(b) Conclude that ℓ^1 is a Banach space.

(c) Where in your proof for (a) did you make use of the fact that you are working with c_0 rather than ℓ^∞?

Solution.

(a) If we restrict the bilinear map from the statement to $\ell^1 \times c_0$, we get a continuous bilinear map

$$\beta \colon \ell^1 \times c_0 \longrightarrow \mathbb{F}.$$

By Exercise 3.25, β_U is linear and continuous. In our notation, this is the function $u \longmapsto u^\vee \colon \ell^1 \longrightarrow \big(c_0\big)^\vee$, where

$$u^\vee(v) = \beta(u,v) = \sum_{n=1}^{\infty} u_n v_n.$$

We have the Hölder Inequality

$$\sum_{n=1}^{\infty} |u_n v_n| \leqslant \|u\|_{\ell^1} \|v\|_{\ell^\infty},$$

valid for all $u \in \ell^1$ and all $v \in \ell^\infty$, so certainly for all $v \in c_0$.

Hence for $v \neq 0$:

$$\frac{|u^\vee(v)|}{\|v\|_{\ell^\infty}} \leqslant \|u\|_{\ell^1},$$

so taking supremum we get $\|u^\vee\| \leqslant \|u\|_{\ell^1}$.

For surjectivity, we need to show that each $\varphi \in \big(c_0\big)^\vee$ is of the form $\varphi = u^\vee$ for some $u \in \ell^1$. Take such φ. Recall that c_0 has Schauder basis $\{e_1, e_2, \dots\}$, so for any $v = (v_n) \in c_0$ we have

$$\varphi(v) = \sum_{n=1}^{\infty} v_n \varphi(e_n).$$

Let $u_n = \varphi(e_n)$ and $u = (u_n)$. We need to show that $u \in \ell^1$. For this, fix $m \in \mathbb{N}$ and let (ignoring the n's for which $u_n = 0$)

$$x = \sum_{n=1}^{m} \frac{|u_n|}{u_n} e_n = \left(\frac{|u_1|}{u_1}, \dots, \frac{|u_m|}{u_m}, 0, 0, \dots \right),$$

so that

$$\|x\|_{\ell^\infty} = 1.$$

Then

$$\begin{aligned}\sum_{n=1}^{m}|u_n| &= \left|\sum_{n=1}^{m}\frac{|u_n|}{u_n}\,u_n\right|\\ &= \left|\sum_{n=1}^{m}\varphi\left(\frac{|u_n|}{u_n}\,e_n\right)\right|\\ &= |\varphi(x)| \leqslant \|\varphi\|\,\|x\|_{\ell^\infty} = \|\varphi\|.\end{aligned}$$

Taking the limit as $m \longrightarrow \infty$ we conclude that $u \in \ell^1$ and that $\|u\|_{\ell^1} \leqslant \|\varphi\|$.

So $u \longmapsto u^\vee$ is surjective.

If we go through the previous construction with $\varphi = u^\vee$, we have $u^\vee(e_n) = \beta(u, e_n) = u_n$, so we land back on u and $\|u\|_{\ell^1} \leqslant \|\varphi\| = \|u^\vee\|$. As we have already established the opposite inequality, we conclude that $\|u^\vee\| = \|u\|_{\ell^1}$, so $u \longmapsto u^\vee$ is distance-preserving.

Putting it all together, we have a linear isometry $\ell^1 \longrightarrow \left(c_0\right)^\vee$.

(b) We know that duals of normed spaces are complete, so $\left(c_0\right)^\vee$ is complete, so ℓ^1, being isometric to it, also is complete.

(c) We used the Schauder basis $\{e_1, e_2, \dots\}$ for c_0 to prove surjectivity as well as the distance-preserving property. $\square$

3. Consider the maps $H_{\text{even}}, H_{\text{odd}} \colon \mathbb{F}^{\mathbb{N}} \longrightarrow \mathbb{F}^{\mathbb{N}}$ defined by

$$H_{\text{even}}\big((a_n)\big) = \big(a_{2n}\big), \qquad H_{\text{odd}}\big((a_n)\big) = \big(a_{2n-1}\big)$$

and construct $f \colon \mathbb{F}^{\mathbb{N}} \longrightarrow \mathbb{F}^{\mathbb{N}} \times \mathbb{F}^{\mathbb{N}}$ as

$$f(a) = \big(H_{\text{even}}(a), H_{\text{odd}}(a)\big).$$

(a) Prove that the restriction of H_{even} and H_{odd} to ℓ^p gives bounded linear functions $H_{\text{even}}, H_{\text{odd}} \colon \ell^p \longrightarrow \ell^p$ for all $p \in \mathbb{R}_{\geqslant 1}$ and for $p = \infty$.

(b) Prove that f is an invertible linear map.

(c) Take $p = 1$ and show that the restriction $f \colon \ell^1 \longrightarrow \ell^1 \times \ell^1$ is a linear isometry.

(Recall that we are working with the norm on $\ell^1 \times \ell^1$ given by

$$\|(x, y)\| := \|x\|_{\ell^1} + \|y\|_{\ell^1}$$

as described in Example 3.4.)

(d) Show that the statement from part (c) does not hold for the space ℓ^∞; prove the strongest statement that you can for ℓ^∞.

(Same comment as in part (c) applies for the norm we consider on $\ell^\infty \times \ell^\infty$.)

Solution. (a) Linearity is straightforward, even on all of $\mathbb{F}^{\mathbb{N}}$:

$$\begin{aligned} H_{\text{even}}\big(\lambda a + \mu b\big) &= H_{\text{even}}\big((\lambda a_n + \mu b_n)\big) \\ &= \big(\lambda a_{2n} + \mu b_{2n}\big) \\ &= \lambda\big(a_{2n}\big) + \mu\big(b_{2n}\big) \\ &= \lambda H_{\text{even}}(a) + \mu H_{\text{even}}(b) \end{aligned}$$

and similarly for H_{odd}.

If $a = (a_n) \in \ell^p$ then

$$\big\|H_{\text{even}}(a)\big\|_{\ell^p}^p = \sum_{n=1}^{\infty} |a_{2n}|^p \leqslant \sum_{n=1}^{\infty} |a_n|^p = \|a\|_{\ell^p}^p,$$

so $H_{\text{even}}(a) \in \ell^p$ and $H_{\text{even}} \colon \ell^p \longrightarrow \ell^p$ is bounded. The same argument works for H_{odd}.

Similarly, if $a = (a_n) \in \ell^\infty$ then

$$\big\|H_{\text{even}}\big\|_{\ell^\infty} = \sup_{n\in\mathbb{N}} |a_{2n}| \leqslant \sup_{n\in\mathbb{N}} |a_n| = \|a\|_{\ell^\infty}$$

and the same for H_{odd}.

(b) The map f is linear because its two components are linear.

We construct an explicit inverse $g \colon \mathbb{F}^{\mathbb{N}} \times \mathbb{F}^{\mathbb{N}} \longrightarrow \mathbb{F}^{\mathbb{N}}$: given $b, c \in \mathbb{F}^{\mathbb{N}}$, define

$$g(b, c) := a := (a_n) \in \mathbb{F}^{\mathbb{N}} \qquad \text{by} \qquad a_n = \begin{cases} b_{n/2} & \text{if } n \text{ is even} \\ c_{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

It is clear that g is the inverse of f.

(c) We have

$$\begin{aligned}
\|f(a)\| &= \left\|(H_{\text{even}}(a), H_{\text{odd}}(a))\right\| \\
&= \left\|H_{\text{even}}(a)\right\|_{\ell^1} + \left\|H_{\text{odd}}(a)\right\|_{\ell^1} \\
&= \sum_{n=1}^{\infty} |a_{2n}| + \sum_{n=1}^{\infty} |a_{2n-1}| \\
&= \sum_{n=1}^{\infty} |a_n| \\
&= \|a\|_{\ell^1},
\end{aligned}$$

so that f is a distance-preserving map.

To prove surjectivity of f, we show that the restriction of the function g from part (b) maps to ℓ^1: for $b, c \in \ell^1$, we have $a := g(b, c)$.

The fact that $a \in \ell^1$ follows from

$$\sum_{n=1}^{2m} |a_n| = \sum_{k=1}^{m} |a_{2k}| + \sum_{k=1}^{m} |a_{2k-1}| = \sum_{k=1}^{m} |b_k| + \sum_{k=1}^{m} |c_k|.$$

As $b, c \in \ell^1$, the limit of the RHS as $m \longrightarrow \infty$ exists and equals $\|b\|_{\ell^1} + \|c\|_{\ell^1}$, so $a \in \ell^1$, $f(a) = (b, c)$, and (of course) $\|a\|_{\ell^1} = \|(b, c)\|$.

(d) We try to use the same approach as in (b):

$$\begin{aligned}
\|f(a)\| &= \left\|(H_{\text{even}}(a), H_{\text{odd}}(a))\right\| \\
&= \left\|H_{\text{even}}(a)\right\|_{\ell^\infty} + \left\|H_{\text{odd}}(a)\right\|_{\ell^\infty} \\
&= \sup_{n\in\mathbb{N}} |a_{2n}| + \sup_{n\in\mathbb{N}} |a_{2n-1}| \\
&\leqslant \sup_{n\in\mathbb{N}} |a_n| + \sup_{n\in\mathbb{N}} |a_n| \\
&= 2\|a\|_{\ell^\infty},
\end{aligned}$$

which shows that f is bounded.

It also indicates that f is not distance-preserving: take $(a) = (1, 1, \dots)$ then

$$\|f(a)\| = 2 \neq 1 = \|a\|_{\ell^\infty}.$$

So far we know that f is linear and bounded. It is also injective because it is the restriction of the injective map from part (b).

To prove surjectivity, we show that the restriction of the function g from part (b) maps to ℓ^∞: for $b, c \in \ell^\infty$, we have $a := g(b, c)$. But

$$\sup_{n\in\mathbb{N}} |a_n| = \sup\left\{\sup_{n\in\mathbb{N}} |a_{2n}|, \sup_{n\in\mathbb{N}} |a_{2n-1}|\right\} = \sup\left\{\|b\|_{\ell^\infty}, \|c\|_{\ell^\infty}\right\},$$

which is finite because it is the maximum of two finite quantities.

Finally, the last equation tells us that

$$\|g(b, c)\| = \|a\| = \sup\left\{\|b\|_{\ell^\infty}, \|c\|_{\ell^\infty}\right\} \leqslant \|b\|_{\ell^\infty} + \|c\|_{\ell^\infty} = \|(b, c)\|,$$

so g is also a bounded function, hence continuous.

We conclude that f is a linear homeomorphism. $\square$

4. Consider the map $f\colon \ell^1 \longrightarrow \mathbb{F}^{\mathbb{N}}$ given by
$$f\big((a_n)\big) = \left(\frac{a_n}{n}\right).$$

(a) Prove that f maps to ℓ^1 and $f\colon \ell^1 \longrightarrow \ell^1$ is linear, continuous, and injective.

(b) Prove that the image W of f is not closed in ℓ^1.

Solution. (a) For all $n \in \mathbb{N}$ we have
$$\left|\frac{a_n}{n}\right| \leqslant |a_n|,$$
so that for $m \in \mathbb{N}$:
$$\sum_{n=1}^{m}\left|\frac{a_n}{n}\right| \leqslant \sum_{n=1}^{m}|a_n|.$$
As $(a_n) \in \ell^1$, the RHS has a finite limit as $m \longrightarrow \infty$, hence so does the LHS, so $f\big((a_n)\big) \in \ell^1$.

Linearity is clear:
$$f\big(\lambda(a_n)+\mu(b_n)\big) = f\big((\lambda a_n+\mu b_n)\big) = \left(\frac{\lambda a_n+\mu b_n}{n}\right) = \lambda\left(\frac{a_n}{n}\right)+\mu\left(\frac{b_n}{n}\right) = \lambda f\big((a_n)\big)+\mu f\big((b_n)\big).$$

We've seen already that $\big\|f\big((a_n)\big)\big\|_{\ell^1} \leqslant \big\|(a_n)\big\|_{\ell^1}$, so f is bounded, hence continuous.

Suppose $f\big((a_n)\big) = f\big((b_n)\big)$, then for all $n \in \mathbb{N}$ we have $a_n/n = b_n/n$, therefore $a_n = b_n$. So f is injective.

(b) For each $n \in \mathbb{N}$ let $v_n = \big(1, 1/2, \ldots, 1/n, 0, 0, \ldots\big) \in \mathbb{F}^{\mathbb{N}}$. Since v_n has only finitely many nonzero terms, it is in ℓ^1. Letting $w_n = f(v_n)$, we have $w_n \in W$.

Set
$$w = \left(1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots\right).$$
Since
$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$
converges, we have $w \in \ell^1$.

However, $w \notin W$: if $w \in W$ then $w = f(v)$ where $v = (1, 1, \ldots)$, but $v \notin \ell^1$.

Finally
$$\|w - w_n\|_{\ell^1} = \left\|\left(0, 0, \ldots, 0, \frac{1}{(n+1)^2}, \frac{1}{(n+2)^2}, \ldots\right)\right\|_{\ell^1} = \sum_{k=n+1}^{\infty}\frac{1}{k^2},$$
which is the tail of a convergent series, hence converges to 0. Therefore $(w_n) \longrightarrow w$, but $w \notin W$, so W is not closed in ℓ^1. $\square$

5. Let $V = \mathbb{R}^2$ viewed as a normed space with the Euclidean norm. Compute the norm of each of the following elements $M \in B(V,V)$ directly from the description of the operator norm:

$$\|M\| = \sup_{\|v\|=1} \|M(v)\|.$$

   (a) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$;

   (b) $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

   (c) $C = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ for $a, b \in \mathbb{R}$.

Solution. In all cases we will denote $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ with $x_1^2 + x_2^2 = 1$.

(a) We have

$$\|Av\| = \left\| \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \right\| = |x_2|.$$

Maximising this under the constraint $x_1^2 + x_2^2 = 1$ gives $\|A\| = 1$.

(b) We have

$$\|Bv\| = \left\| \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \right\| = \sqrt{x_2^2 + x_1^2} = 1,$$

so $\|B\| = 1$.

(c) We have

$$\|Cv\| = \left\| \begin{pmatrix} ax_1 \\ bx_2 \end{pmatrix} \right\| = \sqrt{a^2x_1^2 + b^2x_2^2},$$

so we are looking to maximise, under the constraint $x_1^2 + x_2^2 = 1$, the quantity

$$S = a^2x_1^2 + b^2x_2^2 = a^2x_1^2 + b^2\left(1 - x_1^2\right) = b^2 + (a^2 - b^2)x_1^2.$$

If $|a| \geqslant |b|$ then $a^2 - b^2 \geqslant 0$ so to maximise S we must maximise x_1^2, which happens when $x_1^2 = 1$, so that $S = a^2$.

Otherwise we have $|a| < |b|$ so $a^2 - b^2 < 0$ so to maximise S we must minimise x_1^2, which happens when $x_1 = 0$, so that $S = b^2$.

Hence the maximum value of S is $S = \max\left\{a^2, b^2\right\}$ and so $\|C\| = \sqrt{S} = \max\left\{|a|, |b|\right\}$. $\square$

6. We explore the Hilbert Projection Theorem when V is a Banach space but not a Hilbert space.

   (a) Let $V = \mathbb{R}^2$ with the ℓ^1-norm, that is

   $$\|(x_1, x_2)\| = |x_1| + |x_2|.$$

   Let $Y = \mathbb{B}_1(0)$, the closed unit ball around 0. Find two distinct closest points in Y to $x = (-1, 1) \in V$.

   (b) Can you find a similar example for $V = \mathbb{R}^2$ with the ℓ^∞-norm:

   $$\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}?$$

   (c) Let V be a normed space and Y a convex subset of V. Fix $x \in V$. Let $Z \subseteq Y$ be the set of all closest points in Y to x. Prove that Z is convex.

Solution.

(a) Let $y = (y_1, y_2) \in Y$, then $d(y, 0) \leqslant 1$.

Note that $d(x, 0) = 2$. By the triangle inequality

$$d(x, y) + d(y, 0) \geqslant d(x, 0) \Rightarrow d(x, y) \geqslant d(x, 0) - d(y, 0) \geqslant 2 - 1 = 1.$$

Since this holds for all $y \in Y$, we have $d_Y(x) \geqslant 1$.

But there are (uncountably many) points of Y at distance 1 from x: take any point $y = (y_1, y_2)$ on the line segment joining $(-1, 0)$ to $(0, 1)$, then $y_2 = y_1 + 1$ with $-1 \leqslant y_1 \leqslant 0$ and

$$d(x, y) = |-1 - y_1| + |y_1| = 1 + y_1 - y_1 = 1.$$

We conclude that $d_Y(x) = 1$ and all the points on that line segment are closest points to x.

(b) We can recreate a similar scenario for the ℓ^∞-norm on $V = \mathbb{R}^2$ by taking $Y = \mathbb{B}_1(0)$ and $x = (2, 0)$, for instance.

The same argument as in (a) gives us $d_Y(x) = 1$ and every point on the line segment joining $(1, -1)$ to $(1, 1)$ is at this distance from x.

(c) (Let's note that the conclusion definitely holds for parts (a) and (b), as well as in the Hilbert case covered by the Projection Theorem.)

Let $D = d_Y(x)$.

If Z is empty it is certainly convex.

Otherwise let $z_1, z_2 \in Z$ and let $a \in [0, 1]$. Consider $y = az_1 + (1-a)z_2$. Since $z_1, z_2 \in Z \subseteq Y$ and Y is convex, we have that $y \in Y$. We have

$$\begin{aligned} d(y, x) = \|y - x\| = \|az_1 + (1-a)z_2 - x\| &= \|az_1 - ax + (1-a)z_2 - (1-a)x\| \\ &= \|a(z_1 - x) + (1-a)(z_2 - x)\| \leqslant \|a(z_1 - x)\| + \|(1-a)(z_2 - x)\| \\ &= a\|z_1 - x\| + (1-a)\|z_2 - x\| = aD + (1-a)D = D. \end{aligned}$$

So $d(y, x) \leqslant D$, but also $d(y, x) \geqslant D = d_Y(x)$, so we must have $d(y, x) = D$ and $y \in Z$. □

7. Let $H = \ell^2$ over $\mathbb{R}$ and consider the subset

$$W = \{y = (y_n) \in \ell^2 : y_n \geqslant 0 \text{ for all } n \in \mathbb{N}\}.$$

(a) Prove that W is a closed, convex subset of H. Is it a vector subspace?

(b) Find the closest point $y_{\min} \in W$ to

$$x = (x_n) = \left(\frac{(-1)^n}{n}\right) = \left(-1, \frac{1}{2}, -\frac{1}{3}, \dots\right)$$

and compute $d_W(x)$.

[*Hint*: You may use without proof the identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.]

Solution.

(a) If $y, z \in W$ and $a \in [0,1]$ then $ay + (1-a)z = \big(ay_n + (1-a)z_n\big)$ and it is clear that $ay_n + (1-a)z_n \geqslant 0$, so W is convex.

To show that W is closed we note that

$$W = \bigcap_{n \in \mathbb{N}} \pi_n^{-1}\big([0,\infty)\big),$$

where $\pi_n : \ell^2 \longrightarrow \mathbb{R}$ is given by $\pi_n\big((a_n)\big) = a_n$. We've seen in Exercise 3.19 that π_n is continuous, so since $[0,\infty)$ is closed in $\mathbb{R}$, W is the intersection of a collection of closed subsets, hence it is closed.

Not a vector subspace because not closed under multiplication by $-1 \in \mathbb{R}$.

(b) Let $y = (y_n) \in W$, then

$$\begin{aligned}
\|x - y\|^2 &= \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n} - y_n\right|^2 \\
&= \sum_{n \text{ odd}} \left|-\frac{1}{n} - y_n\right|^2 + \sum_{n \text{ even}} \left|\frac{1}{n} - y_n\right|^2 \\
&= \sum_{n \text{ odd}} \left|\frac{1}{n} + y_n\right|^2 + \sum_{n \text{ even}} \left|\frac{1}{n} - y_n\right|^2
\end{aligned}$$

Note that since $y_n \geqslant 0$:

$$\begin{aligned}
&\text{if } n \text{ is odd then} \quad \left|\frac{1}{n} + y_n\right|^2 \geqslant \frac{1}{n^2} \\
&\text{if } n \text{ is even then} \quad \left|\frac{1}{n} - y_n\right|^2 \geqslant 0.
\end{aligned}$$

Putting this together with the previous result, we get

$$d(x,y)^2 = \|x - y\|^2 \geqslant \sum_{n \text{ odd}} \frac{1}{n^2}.$$

As this holds for all $y \in W$, we get that

$$d_W(x) \geqslant \sqrt{\sum_{n \text{ odd}} \frac{1}{n^2}}.$$

But following the calculations above it is easy to put together an element $y_{\min} = (y_n) \in W$ that achieves this lower bound:

$$y_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Finally, to compute $d_W(x)$, note

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n \text{ even}} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8},$$

hence

$$d_W(x) = \frac{\pi}{2\sqrt{2}}.$$

□