Assignment 1

Note: Due Friday 1 September at 20:00 on Canvas & Gradescope. Please read the instructions given on Canvas. The questions have varying lengths and do not all count for the same number of marks; you may assume that longer questions are worth more.

1.

(a) Let (X, d) be a metric space with X a finite set. Prove that d is equivalent to the discrete metric on X.

(b) Let X be a set and let d be the discrete metric on X.

Is X (i) complete? (ii) compact? (iii) connected? (iv) bounded?

For each property listed, either give a proof that all discrete metric spaces X have the property, or give a specific counterexample of a discrete metric space X that does not have the property.

Solution.

(a) If X is empty, then d equals to the discrete metric vacuously and of course d is equivalent to the discrete metric. If X is not empty, then we prove that every singleton $\{x\}$ is open with respect to the metric d in two cases:

- If X has only one element, then $X = \{x\}$ is open by Example 2.8.
- If X has more than one element, then there are finitely many pairs $(x, y) \in X \times X$, so we can look at the non-empty set

$$\big\{d(x, y) \colon x, y \in X, x \neq y\big\}.$$

The minimum r of this set is the minimum of finitely many positive numbers, hence $r > 0$. Then $\mathbb{B}_r(x) = \{x\}$ for all $x \in X$, showing that $\{x\}$ is open for all $x \in X$.

(b)

(i) Since d is the discrete metric, $d(x, y) = 1$ iff $x \neq y$, so the only Cauchy sequences are the eventually constant sequences of the form $(x_1, \dots, x_n, x_n, x_n, \dots)$, which converges to $x_n \in X$. So yes, X is complete.

(ii) $\mathbb{Z}$ has the open cover

$$\mathbb{Z} = \bigcup_{m \in \mathbb{Z}} \mathbb{B}_1(m),$$

where all the open sets are disjoint, so there is no finite subcover.

In fact, a discrete metric space X is compact iff X is a finite set.

(iii) Let $X = \{x, y\}$ where $x \neq y$. Then $\{x\}$ and $\{y\}$ are open sets and express X as a nontrivial disjoint union of two open sets. So X is disconnected.

In fact, the only connected discrete metric spaces are the empty set and the singletons.

(iv) X is bounded, since $d(x, y) \leqslant 1$ for all $x, y \in X$. □

2. Let X be a set and let d_1, d_2 be two metrics on X.

   (a) Suppose that there exist $m, M \in \mathbb{R}_{>0}$ such that

   $$m\, d_1(x,y) \leqslant d_2(x,y) \leqslant M\, d_1(x,y) \qquad \text{for all } x, y \in X. \tag{1}$$

   Show that d_1 and d_2 are equivalent.

   (b) Prove that the converse of (a) does not hold.

   In other words, find a set X and two equivalent metrics d_1 and d_2 with the property that there **do not** exist positive real numbers m and M such that Equation (1) holds.

Solution.

(a) Use Proposition 2.27. Consider an open ball $\mathbb{B}_r^{d_2}(x)$ of (X, d_2). I claim that the open ball $\mathbb{B}_{r/M}^{d_1}(x)$ of (X, d_1) is contained in $\mathbb{B}_r^{d_2}(x)$: if $y \in \mathbb{B}_{r/M}^{d_1}(x)$ then $d_1(x,y) < r/M$, so that

$$d_2(x,y) \leqslant M\, d_1(x,y) < r.$$

So d_1 is finer than d_2.

Now consider an open ball $\mathbb{B}_r^{d_1}(x)$ of (X, d_1). I claim that the open ball $\mathbb{B}_{rm}^{d_2}(x)$ of (X, d_2) is contained in $\mathbb{B}_r^{d_1}(x)$: if $y \in \mathbb{B}_{rm}^{d_2}(x)$ then $d_2(x,y) < rm$, so that

$$d_1(x,y) \leqslant \frac{1}{m} d_2(x,y) < r.$$

So d_2 is finer than d_1.

(b) Let $X = \mathbb{Z}$. Let d_1 be the discrete metric on $\mathbb{Z}$. Let d_2 be the induced Euclidean metric from $\mathbb{R}$, that is $d_2(x,y) = |x - y|$ for all $x, y \in \mathbb{Z}$.

First we note that d_1 and d_2 are equivalent metrics. It suffices to show that every singleton $\{x\} \subseteq \mathbb{Z}$ is open with respect to d_2:

$$\mathbb{B}_1^{d_2}(x) = \{y \in \mathbb{Z} \colon |y - x| < 1\} = \{y \in \mathbb{Z} \colon x - 1 < y < x + 1\} = \{x\}.$$

Suppose that d_1 and d_2 satisfy Equation (1) for some $m, M > 0$. In particular, if $x \neq y$ we would have

$$m \leqslant |x - y| \leqslant M \qquad \text{for all } x \neq y \in \mathbb{Z},$$

which is blatantly false (take $y = 0$, $x = \lceil M \rceil + 1$). □

3. Let X be a compact metric space and $\{C_i : i \in I\}$ be a collection of closed subsets of X such that

$$\bigcap_{j \in J} C_j \neq \varnothing \qquad \text{for every finite subset } J \subseteq I.$$

Prove that

$$\bigcap_{i \in I} C_i \neq \varnothing.$$

Give an example showing that the conclusion need not hold without the compactness condition.

Solution. Suppose that

$$\bigcap_{i \in I} C_i = \varnothing.$$

Therefore

$$X = \bigcup_{i \in I} U_i, \qquad \text{where } U_i := X \setminus C_i,$$

is an open covering of X. Since X is compact, there exists a finite subset $J \subseteq I$ such that

$$X = \bigcup_{j \in J} U_j,$$

which implies that

$$\bigcup_{j \in J} C_j = \varnothing,$$

contradicting the hypothesis on the collection $\{C_i : i \in I\}$.

Here is a counterexample where X is not compact. Take $X = \mathbb{R}_{>0}$, $I = \mathbb{N}$, and $C_i = (0, 1/i]$ for $i \in I$. Then each C_i is closed in X: since both X and $(1/i, \infty)$ are open in $\mathbb{R}$, we conclude that $X \setminus C_i = X \cap (1/i, \infty)$ is open in X.

Also,

$$\bigcap_{i \in I} C_i = \varnothing,$$

because if $x \in \mathbb{R}_{>0}$ is in C_i for all $i \in I$, then $0 < x \leqslant 1/i$ for all i in I, hence $0 < x \leqslant 0$ by taking limits as $i \longrightarrow \infty$, contradiction.

If $J \subseteq I$ is finite, let $m = \max\{J\}$, then

$$\bigcap_{j \in J} C_j = C_m \neq \varnothing. \qquad \square$$

4. Let (X, d) be a metric space and define $d' \colon X \times X \longrightarrow \mathbb{R}_{\geqslant 0}$ by

$$d'(x, y) = \min\left\{d(x, y), 1\right\}.$$

Prove that d' is a metric on X and that d' is equivalent to d.

Solution. It is clear that $d'(y, x) = d'(x, y)$ and that $d'(x, y) = 0$ if and only if $d(x, y) = 0$ if and only if $x = y$.

For the triangle inequality: $d'(x, y) \leqslant 1$ so if at least one of $d'(x, t)$, $d'(t, y)$ is 1, the triangle inequality holds. So we may assume that $d'(x, t) = d(x, t)$ and $d'(t, y) = d(t, y)$. Then

$$d'(x, y) \leqslant d(x, y) \leqslant d(x, t) + d(t, y) = d'(x, t) + d'(t, y).$$

It remains to prove the equivalence of d and d'. Let $x \in X$ and $s \leqslant 1$.

I claim that $\mathbb{B}^d_s(x) = \mathbb{B}^{d'}_s(x)$. To see this, let $y \in \mathbb{B}^d_s(x)$, then $d(x, y) < s \leqslant 1$, so

$$d'(x, y) = \min\{d(x, y), 1)\} = d(x, y) < s.$$

In the other direction, let $y \in \mathbb{B}^{d'}_s(x)$, then

$$\min\{d(x, y), 1\} = d'(x, y) < s \leqslant 1,$$

which forces $d(x, y) = d'(x, y) < s$.

We conclude by noting that for any $r > 0$, if we set $s = \min\{r, 1\}$ we get $\mathbb{B}^d_s(x) = \mathbb{B}^{d'}_s(x) \subseteq \mathbb{B}^{d'}_r(x)$, and $\mathbb{B}^{d'}_s(x) = \mathbb{B}^d_s(x) \subseteq \mathbb{B}^d_r(x)$. In other words, any d'-open ball contains a d-open ball, and vice-versa. $\square$

5. Let (X, d) be a metric space.

   (a) Fix an arbitrary element $y \in X$ and consider the function $f\colon X \longrightarrow \mathbb{R}$ given by $f(x) = d(x, y)$. Prove that f is uniformly continuous.

   (b) Give $X \times X$ any conserving metric D coming from d. Prove that $d\colon X \times X \longrightarrow \mathbb{R}$ is uniformly continuous (with respect to D).

   (c) Let d' be a metric on X and put on $X \times X$ any conserving metric D' coming from d'. Suppose that $d\colon X \times X \longrightarrow \mathbb{R}$ is continuous with respect to D'. Prove that d' is a finer metric than d.

Solution.

(a) Let $\varepsilon > 0$. Set $\delta = \varepsilon$. If $x, x' \in X$ satisfy $d(x, x') < \delta = \varepsilon$, then

$$|f(x) - f(x')| = |d(x, y) - d(x', y)| \leqslant d(x, x') < \varepsilon.$$

(b) Let $\varepsilon > 0$. Set $\delta = \varepsilon/2$. If $(x_1, x_2), (x_1', x_2') \in X \times X$ satisfy

$$\max\{d(x_1, x_1'), d(x_2, x_2')\} \leqslant D((x_1, x_2), (x_1', x_2')) < \delta = \frac{\varepsilon}{2},$$

(where we used the fact that D is conserving), then

$$|d(x_1, x_2) - d(x_1', x_2')| \leqslant d(x_1, x_1') + d(x_2, x_2') < \varepsilon,$$

where the first inequality is obtained by applying the triangle inequality a couple of times, as in Example 2.37.

(c) We prove that if $(x_n) \longrightarrow x \in X$ with respect to d', then $(x_n) \longrightarrow x$ with respect to x.

Suppose $(x_n) \longrightarrow x \in X$ with respect to d'. Then $\big((x_n, x)\big) \longrightarrow (x, x) \in X \times X$ with respect to D'. But $d\colon X \times X \longrightarrow \mathbb{R}$ is continuous with respect to D', so $(d(x_n, x)) \longrightarrow d(x, x) = 0 \in \mathbb{R}$. Therefore $(x_n) \longrightarrow x \in X$ with respect to d. □

6. Give $\mathbb{Q} \subseteq \mathbb{R}$ the induced metric and consider the sequence (x_n) defined recursively by

$$x_1 = 1, \qquad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \quad \text{for } n \in \mathbb{N}.$$

   (a) Prove that $1 \leqslant x_n \leqslant 2$ for all $n \in \mathbb{N}$ and breathe a sigh of relief that the recursive definition does not accidentally divide by 0.

   (b) For $n \in \mathbb{N}$, let $y_n = x_{n+1} - x_n$. Prove that

$$y_{n+1} = -\frac{y_n^2}{2x_{n+1}} \qquad \text{for all } n \in \mathbb{N}.$$

   (c) Prove that

$$|y_n| \leqslant \frac{1}{2^n} \qquad \text{for all } n \in \mathbb{N}.$$

   (d) Show that (x_n) is Cauchy.

   (e) Consider the function $f\colon [1,2] \longrightarrow [1,2]$ given by

$$f(x) = \frac{x}{2} + \frac{1}{x}.$$

   Prove that f is a contraction. What is the fixed point of f?

Solution.

(a) Induction on n. Base case $x_1 = 1$ clear.

Fix $n \in \mathbb{N}$ and suppose $1 \leqslant x_n \leqslant 2$. Then

$$\frac{1}{2} \leqslant \frac{x_n}{2} \leqslant 1 \qquad \text{and} \qquad \frac{1}{2} \leqslant \frac{1}{x_n} \leqslant 1,$$

so $1 \leqslant x_{n+1} \leqslant 2$.

(b) Fix $n \in \mathbb{N}$. Noting that $2x_n x_{n+1} = x_n^2 + 2$, we have

$$y_n^2 = \left(x_{n+1} - x_n\right)^2 = x_{n+1}^2 - 2x_{n+1}x_n + x_n^2 = x_{n+1}^2 - 2$$
$$2x_{n+1}y_{n+1} = 2x_{n+1}\left(\frac{1}{x_{n+1}} - \frac{x_{n+1}}{2}\right) = 2 - x_{n+1}^2 = -y_n^2.$$

(c) From part (b) we have

$$|y_{n+1}| = \frac{|y_n|^2}{2x_{n+1}} \qquad \text{for all } n \in \mathbb{N}.$$

We can use this, part (a), and induction by n.

For the base case we have $y_1 = \frac{1}{2}$.

For the induction step, fix $n \in \mathbb{N}$ and suppose $|y_n| \leqslant \frac{1}{2^n}$, then

$$|y_{n+1}| = \frac{|y_n|^2}{2x_{n+1}} \leqslant \frac{|y_n|^2}{2} \leqslant \frac{1}{2^{2n+1}} \leqslant \frac{1}{2^{n+1}}.$$

(d) Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that $2^N > 1/\varepsilon$. If $n \geqslant m \geqslant N$ then

$$\begin{aligned}
|x_n - x_m| &= |y_{n-1} + y_{n-2} + \cdots + y_{m+1}| \\
&\leqslant |y_{n-1}| + \cdots + |y_{m+1}| \\
&\leqslant \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^{m+1}} \\
&= \left(\frac{1}{2^{n-m-2}} + \frac{1}{2^{n-m-3}} + \cdots + 1 \right) \frac{1}{2^{m+1}} \\
&\leqslant \frac{2}{2^{m+1}} \leqslant \frac{1}{2^N} < \varepsilon.
\end{aligned}$$

Here we used the fact that the geometric series with ratio 1/2 sums up to 2.

(e) Let $x_1, x_2 \in [1,2]$. The function f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), so there exists $\xi \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi),$$

from which we deduce that

$$|f(x_2) - f(x_1)| = |f'(\xi)|\,|x_2 - x_1|.$$

But since $\xi \in (1,2)$ we have

$$1 < \xi < 2 \Rightarrow \frac{1}{4} < \frac{1}{\xi^2} < 1 \Rightarrow -\frac{1}{2} < f'(\xi) < \frac{1}{2}.$$

We conclude that f is a contraction with constant 1/2.

Since f is a contraction and $[1,2]$ is complete, we know that f has a unique fixed point, which is precisely the limit of the sequence (x_n) defined above. We can find it explicitly as

$$x = f(x) = \frac{x}{2} + \frac{1}{x} \Rightarrow x^2 = 2,$$

and since $x \in [1,2]$ we get $x = \sqrt{2}$. □

7. Let $\mathbb{S}^1 = \mathbb{S}_1\big((0,0)\big) = \big\{x, y \in \mathbb{R} \colon x^2 + y^2 = 1\big\}$ be the unit circle in $\mathbb{R}^2$.

   Consider the function $f \colon [0,1) \longrightarrow \mathbb{S}^1$ given by the parametrisation

$$f(t) = \big(\cos(2\pi t), \sin(2\pi t)\big).$$

   Endow $[0,1)$ with the induced metric from $\mathbb{R}$ and $\mathbb{S}^1$ with the induced metric from $\mathbb{R}^2$.

   Prove that f is a bijective continuous function, but not a homeomorphism.

   (You may use without proof whatever properties of the functions sin and cos you manage to remember from previous subjects.)

   Solution.

   (a) We know that $t \longmapsto 2\pi t$, $t \longmapsto \cos(t)$ and $t \longmapsto \sin(t)$ are continuous, so by Exercise 2.29 so is f.

   (b) Suppose $t_1 \neq t_2 \in [0,1)$ are such that $f(t_1) = f(t_2)$. Then $\cos(2\pi t_1) = \cos(2\pi t_2)$, which implies that $t_2 = 1 - t_1$. In that case $\sin(2\pi t_2) = \sin(2\pi - 2\pi t_1) = \sin(-2\pi t_1) = -\sin(2\pi t_1)$. But we also have $\sin(2\pi t_2) = \sin(2\pi t_1)$, so $\sin(2\pi t_1) = 0$, hence $t_1 = 0$ and $t_2 = 1 - t_1 = 1$, contradicting $t_2 \in [0,1)$.

   We conclude that f is injective.

   For surjectivity, let $(x,y) \in \mathbb{S}^1$, in other words $x^2 + y^2 = 1$. Define $\theta \in [0, 2\pi)$ by

$$\theta = \begin{cases} \arccos(x) & \text{if } y \geqslant 0 \\ 2\pi - \arccos(x) & \text{if } y < 0. \end{cases}$$

   Letting $t = \theta/(2\pi)$, we have $f(t) = (x,y)$.

   (c) At this point we know that f is a homeomorphism iff $f^{-1} \colon \mathbb{S}^1 \longrightarrow [0,1)$ is continuous. Note that $\mathbb{S}^1 \subseteq \mathbb{R}^2$ is compact: it is clearly bounded as any two points are at distance at most 2 of each other, so we just need to check that it is a closed subset of $\mathbb{R}^2$.

   But $\mathbb{S}^1 = \mathbb{D}_1\big((0,0)\big) \cap C$ is the intersection of two closed sets, where

$$C = \{x, y \in \mathbb{R} \colon x^2 + y^2 \geqslant 1\} = \mathbb{R}^2 \setminus \mathbb{B}_1\big((0,0)\big).$$

   Since $\mathbb{S}^1$ is compact, if f^{-1} were continuous then $[0,1) = f^{-1}(\mathbb{S}^1)$ would be compact, hence closed in $\mathbb{R}$. This is a contradiction, because 1 is an accumulation point of $[0,1)$ but does not lie in the set. □

8. Let X be a metric space and Y a complete metric space. Let $D \subseteq X$ be a dense subset and $f: D \longrightarrow Y$ a uniformly continuous function.

   (a) Prove that f has a unique uniformly continuous extension to X, that is there exists a unique uniformly continuous function

   $$\widehat{f}: X \longrightarrow Y \quad \text{such that} \quad \widehat{f}(u) = f(u) \quad \text{for all } u \in D.$$

   (Make sure you give a complete argument: how do you construct $\widehat{f}$? is it well-defined? does it extend f? why is it uniformly continuous? why is it unique?)

   (b) If, in addition, f is distance-preserving, then so is the extension $\widehat{f}$.

   (c) Show that any uniformly continuous (resp. distance-preserving) function $g: X \longrightarrow Y$ between arbitrary metric spaces has a unique uniformly continuous (resp. distance-preserving) extension to completions, $\widehat{g}: \widehat{X} \longrightarrow \widehat{Y}$.

Solution.

(a) The first task is to **construct the function** $\widehat{f}: X \longrightarrow Y$. Let $x \in X$. Since D is dense in X, there exists a sequence (u_n) in D such that $(u_n) \longrightarrow x$. In particular, (u_n) is Cauchy in D. Since $f: D \longrightarrow Y$ is uniformly continuous, $\big(f(u_n)\big)$ is Cauchy in Y. As Y is complete, $\big(f(u_n)\big)$ has a limit $y \in Y$.

Define $\widehat{f}(x) = y$.

But wait, is this actually **well-defined**? We did make one choice in the construction, namely a sequence (u_n) in D that converges to x. Any other valid choice is a sequence (u'_n) in D with the same limit x, so $(u'_n) \sim (u_n)$. As f is continuous, we have $\big(f(u'_n)\big) \sim \big(f(u_n)\big)$, which implies that $\big(f(u'_n)\big) \longrightarrow y \in Y$.

Is $\widehat{f}$ an **extension of** f? If $u \in D$ and we work through the above construction, we see that we can take $u_n = u$ for all $n \in \mathbb{N}$, so $f(u_n) = f(u)$ for all $n \in \mathbb{N}$, and finally $\widehat{f}(u) = y = f(u)$. In other words, $\widehat{f}(u) = f(u)$ for $u \in D$, as claimed.

Next we prove **uniform continuity** of $\widehat{f}$. Let $\varepsilon > 0$. Since $f: D \longrightarrow Y$ is uniformly continuous, there exists $\delta > 0$ such that for all $u, u' \in D$, if $d_X(u, u') < \delta$, then $d_Y(f(u), f(u')) < \varepsilon/2$. Now suppose that $x, x' \in X$ satisfy $d_X(x, x') < \delta/3$. Let (u_n) be a sequence as in the definition of $\widehat{f}(x)$ above, and similarly with (u'_n) and $\widehat{f}(x')$. As $(u_n) \longrightarrow x$, there exists $N \in \mathbb{N}$ such that $d_X(u_n, x) < \delta/3$ for all $n \geqslant N$. Similarly, as $(u'_n) \longrightarrow x'$, there exists $N' \in \mathbb{N}$ such that $d_X(u'_n, x') < \delta/3$ for all $n \geqslant N'$. Letting $M = \max\{N, N'\}$ we get for all $n \geqslant M$:

$$d_X(u_n, u'_n) \leqslant d_X(u_n, x) + d_X(x, x') + d_X(x', u'_n) < \delta.$$

Therefore $d_Y(f(u_n), f(u'_n)) < \varepsilon/2$ for all $n \geqslant M$.

As $\widehat{f}(x) = \lim f(u_n)$ and $\widehat{f}(x') = \lim f(u'_n)$, we conclude that

$$d_Y(\widehat{f}(x), \widehat{f}(x')) \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

The **uniqueness** of $\widehat{f}$ follows from Example 2.43, which says that there is at most one continuous extension.

(b) If f is **distance-preserving**, we use the same line of argument, only simpler. Let $(u_n) \longrightarrow x$, $(u'_n) \longrightarrow x'$ with $u_n, u'_n \in D$. Then

$$\begin{aligned} d_Y\big(\widehat{f}(x), \widehat{f}(x')\big) &= d_Y\Big(\lim_{n\longrightarrow\infty} \widehat{f}(u_n), \lim_{n\longrightarrow\infty} \widehat{f}(u'_n)\Big) \\ &= \lim_{n\longrightarrow\infty} d_Y(f(u_n), f(u'_n)) = \lim_{n\longrightarrow\infty} d_X(u_n, u'_n) = d_X(x, x'). \end{aligned}$$

(c) For the case of completions, let $D = \iota(X) \subseteq \widehat{X}$, and apply the above to the function $\iota_Y \circ g \circ \iota_X^{-1} \colon D \longrightarrow \widehat{Y}$. □