Assignment 2: corrected version as of 2 October

Note: Due Friday 13 October at 20:00 on Canvas & Gradescope. Please read the instructions given on Canvas. The questions have varying lengths and do not all count for the same number of marks; you may assume that longer questions are worth more.

1. Let U, V, W be normed spaces over $\mathbb{F}$.

   Suppose $\beta\colon U \times V \longrightarrow W$ is a continuous bilinear map.

   Consider the linear function $\beta_U\colon U \longrightarrow \operatorname{Hom}(V, W)$ given by $\beta_U(u) = f_u$, where

   $$f_u\colon V \longrightarrow W \qquad \text{is defined by } f_u(v) = \beta(u, v).$$

   (a) Prove that for any $u \in U$, $f_u \in B(V, W)$, in other words f_u is continuous.

   (b) By part (a) we can think of β_U as a function $U \longrightarrow B(V, W)$.
   Prove that $\beta_U\colon U \longrightarrow B(V, W)$ is continuous.

2. In Proposition 3.23 we saw that the function

   $$\ell^1 \times \ell^\infty \longrightarrow \mathbb{F} \qquad \text{defined by } (u, v) \longmapsto \sum_{n=1}^{\infty} u_n v_n$$

   is a continuous bilinear map.

   (a) Show that there is a continuous linear function $\ell^1 \longrightarrow (c_0)^\vee$ that is an isometry.
   (Recall that $c_0 \subseteq \ell^\infty$ consists of all convergent sequences with limit 0.)

   [*Hint*: It may be useful to prove surjectivity first, and then the distance-preserving property.]

   (b) Conclude that ℓ^1 is a Banach space.

   (c) Where in your proof for (a) did you make use of the fact that you are working with c_0 rather than ℓ^∞?

3. Consider the maps $H_{\text{even}}, H_{\text{odd}}\colon \mathbb{F}^{\mathbb{N}} \longrightarrow \mathbb{F}^{\mathbb{N}}$ defined by

   $$H_{\text{even}}\big((a_n)\big) = \big(a_{2n}\big), \qquad H_{\text{odd}}\big((a_n)\big) = \big(a_{2n-1}\big)$$

   and construct $f\colon \mathbb{F}^{\mathbb{N}} \longrightarrow \mathbb{F}^{\mathbb{N}} \times \mathbb{F}^{\mathbb{N}}$ as

   $$f(a) = \big(H_{\text{even}}(a), H_{\text{odd}}(a)\big).$$

   (a) Prove that the restriction of H_{even} and H_{odd} to ℓ^p gives bounded linear functions $H_{\text{even}}, H_{\text{odd}}\colon \ell^p \longrightarrow \ell^p$ for all $p \in \mathbb{R}_{\geqslant 1}$ and for $p = \infty$.

   (b) Prove that f is an invertible linear map.

   (c) Take $p = 1$ and show that the restriction $f\colon \ell^1 \longrightarrow \ell^1 \times \ell^1$ is a linear isometry.

   (Recall that we are working with the norm on $\ell^1 \times \ell^1$ given by

   $$\|(x, y)\| := \|x\|_{\ell^1} + \|y\|_{\ell^1}$$

   as described in Example 3.4.)

   (d) Show that the statement from part (c) does not hold for the space ℓ^∞; prove the strongest statement that you can for ℓ^∞.

   (Same comment as in part (c) applies for the norm we consider on $\ell^\infty \times \ell^\infty$.)

4. Consider the map $f\colon \ell^1 \longrightarrow \mathbb{F}^{\mathbb{N}}$ given by

$$f\big((a_n)\big) = \left(\frac{a_n}{n}\right).$$

   (a) Prove that f maps to ℓ^1 and $f\colon \ell^1 \longrightarrow \ell^1$ is linear, continuous, and injective.

   (b) Prove that the image W of f is not closed in ℓ^1.

5. Let $V = \mathbb{R}^2$ viewed as a normed space with the Euclidean norm. Compute the norm of each of the following elements $M \in B(V,V)$ directly from the description of the operator norm:

$$\|M\| = \sup_{\|v\|=1} \|M(v)\|.$$

   (a) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$;

   (b) $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

   (c) $C = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ for $a, b \in \mathbb{R}$.

6. We explore the Hilbert Projection Theorem when V is a Banach space but not a Hilbert space.

   (a) Let $V = \mathbb{R}^2$ with the ℓ^1-norm, that is

$$\big\|(x_1, x_2)\big\| = |x_1| + |x_2|.$$

   Let $Y = \mathbb{B}_1(0)$, the closed unit ball around 0. Find two distinct closest points in Y to $x = (-1, 1) \in V$.

   (b) Can you find a similar example for $V = \mathbb{R}^2$ with the ℓ^∞-norm:

$$\big\|(x_1, x_2)\big\| = \max\big\{|x_1|, |x_2|\big\}?$$

   (c) Let V be a normed space and Y a convex subset of V. Fix $x \in V$. Let $Z \subseteq Y$ be the set of all closest points in Y to x. Prove that Z is convex.

7. Let $H = \ell^2$ over $\mathbb{R}$ and consider the subset

$$W = \big\{y = (y_n) \in \ell^2 \colon y_n \geqslant 0 \text{ for all } n \in \mathbb{N}\big\}.$$

   (a) Prove that W is a closed, convex subset of H. Is it a vector subspace?

   (b) Find the closest point $y_{\min} \in W$ to

$$x = (x_n) = \left(\frac{(-1)^n}{n}\right) = \left(-1, \frac{1}{2}, -\frac{1}{3}, \dots\right)$$

   and compute $d_W(x)$.

   [*Hint*: You may use without proof the identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.]