Assignment 1

Note: Due Friday 1 September at 20:00 on Canvas & Gradescope. Please read the instructions given on Canvas. The questions have varying lengths and do not all count for the same number of marks; you may assume that longer questions are worth more.

1.
   (a) Let (X, d) be a metric space with X a finite set. Prove that d is equivalent to the discrete metric on X.

   (b) Let X be a set and let d be the discrete metric on X.

   Is X (i) complete? (ii) compact? (iii) connected? (iv) bounded?

   For each property listed, either give a proof that all discrete metric spaces X have the property, or give a specific counterexample of a discrete metric space X that does not have the property.

2. Let X be a set and let d_1, d_2 be two metrics on X.

   (a) Suppose that there exist $m, M \in \mathbb{R}_{>0}$ such that

$$m\, d_1(x,y) \leqslant d_2(x,y) \leqslant M\, d_1(x,y) \qquad \text{for all } x, y \in X. \tag{1}$$

   Show that d_1 and d_2 are equivalent.

   (b) Prove that the converse of (a) does not hold.

   In other words, find a set X and two equivalent metrics d_1 and d_2 with the property that there **do not** exist positive real numbers m and M such that Equation (1) holds.

3. Let X be a compact metric space and $\{C_i : i \in I\}$ be a collection of closed subsets of X such that

$$\bigcap_{j \in J} C_j \neq \varnothing \qquad \text{for every finite subset } J \subseteq I.$$

   Prove that

$$\bigcap_{i \in I} C_i \neq \varnothing.$$

   Give an example showing that the conclusion need not hold without the compactness condition.

4. Let (X, d) be a metric space and define $d' : X \times X \longrightarrow \mathbb{R}_{\geqslant 0}$ by

$$d'(x,y) = \min\{d(x,y), 1\}.$$

   Prove that d' is a metric on X and that d' is equivalent to d.

5. Let (X, d) be a metric space.

   (a) Fix an arbitrary element $y \in X$ and consider the function $f : X \longrightarrow \mathbb{R}$ given by $f(x) = d(x, y)$. Prove that f is uniformly continuous.

   (b) Give $X \times X$ any conserving metric D coming from d. Prove that $d : X \times X \longrightarrow \mathbb{R}$ is uniformly continuous (with respect to D).

   (c) Let d' be a metric on X and put on $X \times X$ any conserving metric D' coming from d'. Suppose that $d : X \times X \longrightarrow \mathbb{R}$ is continuous with respect to D'. Prove that d' is a finer metric than d.

6. Give $\mathbb{Q} \subseteq \mathbb{R}$ the induced metric and consider the sequence (x_n) defined recursively by

$$x_1 = 1, \qquad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \quad \text{for } n \in \mathbb{N}.$$

   (a) Prove that $1 \leqslant x_n \leqslant 2$ for all $n \in \mathbb{N}$ and breathe a sigh of relief that the recursive definition does not accidentally divide by 0.

   (b) For $n \in \mathbb{N}$, let $y_n = x_{n+1} - x_n$. Prove that

$$y_{n+1} = -\frac{y_n^2}{2x_{n+1}} \qquad \text{for all } n \in \mathbb{N}.$$

   (c) Prove that

$$|y_n| \leqslant \frac{1}{2^n} \qquad \text{for all } n \in \mathbb{N}.$$

   (d) Show that (x_n) is Cauchy.

   (e) Consider the function $f\colon [1,2] \longrightarrow [1,2]$ given by

$$f(x) = \frac{x}{2} + \frac{1}{x}.$$

   Prove that f is a contraction. What is the fixed point of f?

7. Let $\mathbb{S}^1 = \mathbb{S}_1\big((0,0)\big) = \big\{x, y \in \mathbb{R}\colon x^2 + y^2 = 1\big\}$ be the unit circle in $\mathbb{R}^2$.

   Consider the function $f\colon [0,1) \longrightarrow \mathbb{S}^1$ given by the parametrisation

$$f(t) = \big(\cos(2\pi t), \sin(2\pi t)\big).$$

   Endow $[0,1)$ with the induced metric from $\mathbb{R}$ and $\mathbb{S}^1$ with the induced metric from $\mathbb{R}^2$.

   Prove that f is a bijective continuous function, but not a homeomorphism.

   (You may use without proof whatever properties of the functions sin and cos you manage to remember from previous subjects.)

8. Let X be a metric space and Y a complete metric space. Let $D \subseteq X$ be a dense subset and $f\colon D \longrightarrow Y$ a uniformly continuous function.

   (a) Prove that f has a unique uniformly continuous extension to X, that is there exists a unique uniformly continuous function

$$\widehat{f}\colon X \longrightarrow Y \quad \text{such that} \quad \widehat{f}(u) = f(u) \quad \text{for all } u \in D.$$

   (Make sure you give a complete argument: how do you construct $\widehat{f}$? is it well-defined? does it extend f? why is it uniformly continuous? why is it unique?)

   (b) If, in addition, f is distance-preserving, then so is the extension $\widehat{f}$.

   (c) Show that any uniformly continuous (resp. distance-preserving) function $g\colon X \longrightarrow Y$ between arbitrary metric spaces has a unique uniformly continuous (resp. distance-preserving) extension to completions, $\widehat{g}\colon \widehat{X} \longrightarrow \widehat{Y}$.