THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assessment, 2021

School of Mathematics and Statistics

MAST30026 Metric and Hilbert Spaces

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 2 pages (including this page) with 5 questions and 100 total marks

Permitted Materials

- This exam and/or an offline electronic PDF reader and blank loose-leaf paper.
- No books, notes or other printed or handwritten material are permitted.
- No calculators are permitted. No headphones or earphones are permitted.

Instructions to Students

- Wave your hand right in front of your webcam if you wish to communicate with the supervisor at any time (before, during or after the exam).
- You must not be out of webcam view at any time without supervisor permission.
- You must not write your answers on an iPad or other electronic device.
- Off-line PDF readers (i) must have the screen visible in Zoom; (ii) must only be used to read exam questions (do not access other software or files); (iii) must be set in flight mode or have both internet and Bluetooth disabled as soon as the exam paper is downloaded.

Writing

- Write your answers on A4 paper. Page 1 should only have your student number, the subject code and the subject name. Write on one side of each sheet only. Each question should be on a new page. The question number must be written at the top of each page.

Scanning and Submitting

- **You must not leave Zoom supervision to scan your exam.** Put the pages in question order and all the same way up. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4.
- Submit your scanned exam as a single PDF file and carefully review the submission in Gradescope. Scan again and resubmit if necessary. Do not leave Zoom supervision until you have confirmed orally with the supervisor that you have received the Gradescope confirmation email.
- **You must not submit or resubmit after having left Zoom supervision.**

Question 1 (20 marks)

(a) State the definition of compactness for topological spaces.

(b) Prove that if X, Y are topological spaces and $f : X \longrightarrow Y$ is a continuous function, then for any compact subset $K \subseteq X$ the image $f(K) \subseteq Y$ is compact.

(c) Prove the Extreme Value Theorem for continuous maps: if f is a continuous real-valued function on a nonempty compact topological space X then there exist $c, d \in X$ such that $f(c) \geq f(x) \geq f(d)$ for all $x \in X$ (you may assume without proof that compact subsets of $\mathbb{R}$ are closed and bounded).

Question 2 (20 marks) Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \longrightarrow Y$ a function. Prove that f is continuous (in the sense that preimages of open sets are open) **if and only if** it is continuous in the ε-δ sense, that is, for all $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ with the property that whenever $y \in X$ and $d_X(x, y) < \delta$ we have $d_Y(f(x), f(y)) < \varepsilon$.

Question 3 (20 marks) Let X be a Hausdorff topological space.

(a) Prove that if $K \subseteq X$ is compact then it is closed.

(b) Prove that if X is compact then it is normal.

Question 4 (20 marks) Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$.

(a) Let V be an inner product space over $\mathbb{F}$, and prove that for $u \in V$ the function $\langle -, u \rangle : V \longrightarrow \mathbb{F}$ is bounded, linear and has operator norm $\|u\|$.

(b) Let H be a Hilbert space over $\mathbb{F}$. If $K \subseteq H$ is closed, convex and nonempty, prove that for each $h \in H$ there is a unique point $k \in K$ which is closest to h, in the sense that

$$\|h - k\| = \inf\{\|h - v\| \mid v \in K\}\,.$$

(Hint: recall the Parallelogram Law $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$).

Question 5 (20 marks) Let (X, d) be a compact metric space. Given a real number $0 < \lambda < 1$ a function $f : X \longrightarrow X$ is called a λ-*contraction* if $d(fx, fy) \leq \lambda d(x, y)$ for all $x, y \in X$. Let $\mathrm{Cts}_\lambda(X, X) \subseteq \mathrm{Cts}(X, X)$ be the subset of λ-contractions with the subspace topology, where $\mathrm{Cts}(X, X)$ has the compact-open topology. Prove that

(a) The function $\mathcal{F} : \mathrm{Cts}_\lambda(X, X) \longrightarrow X$ sending a λ-contraction to its unique fixed point (which exists by the Banach fixed point theorem) is a **continuous** map.

(b) Let $X \subseteq \mathbb{R}^n$ be a compact and convex subset (recall convexity means that for any $x, y \in X$ we have $\{\alpha x + (1 - \alpha)y \mid \alpha \in [0, 1]\} \subseteq X$). Prove that $\mathcal{F}$ has a continuous right inverse, by defining a function $\mathcal{G} : X \longrightarrow \mathrm{Cts}_\lambda(X, X)$ and proving that

(i) $\mathcal{G}$ is well-defined, that is, $\mathcal{G}(y)$ is a λ-contraction for every $y \in X$,

(ii) $\mathcal{F} \circ \mathcal{G} = 1$,

(iii) $\mathcal{G}$ is continuous.

End of Exam — Total Available Marks = 100