Assignment 2

1. Let $m = p^r$ with p prime and $r \in \mathbb{N}$ and let $n = \varphi(m)$. Let $\zeta = e^{2\pi i/m}$ and $K = \mathbb{Q}(\zeta)$. Show that
$$\Delta(\zeta) = \frac{(-1)^{n/2} m^n}{p^{m/p}}.$$

2. Let K be a number field and consider its embeddings $\sigma_1, \ldots, \sigma_n \colon K \to \mathbb{C}$. Let r_1 denote the number of embeddings whose image is actually contained in $\mathbb{R}$. The remaining $n - r_1$ embeddings come in pairs $\sigma, \overline{\sigma}$, where $\overline{\sigma}$ is the composition of σ and the complex conjugation automorphism of $\mathbb{C}$. Let r_2 be the number of such pairs, so that $n = r_1 + 2r_2$.

   Prove that the sign of Δ_K is $(-1)^{r_2}$.

3. Fix $g, n \in \mathbb{Z}_{>1}$ with n odd such that $d := n^g - 1$ is squarefree. Show that the ideal class group of $\mathbb{Q}(\sqrt{-d})$ contains an element of order equal to g.

4. Find the class number of $\mathbb{Q}(\sqrt{-19})$.

5. Let p be a prime number that is congruent to 13 or 17 modulo 20.
   a) Show that the congruence $x^4 \equiv 25 \pmod{p}$ has no solutions.
   b) Show that the equation $x^4 + py^4 = 25z^4$ has no integer solutions other than $(0, 0, 0)$.

6. Let $K = \mathbb{Q}(\sqrt{-6})$. Determine which prime numbers p split, ramify, respectively remain inert in K, expressing your answer in terms of congruence conditions on p.