Assignment 1

1. Let R be a Dedekind domain and let $I \neq 0$ be an ideal of R.
   (a) If $I = \mathfrak{p}_1 \dots \mathfrak{p}_n$ is the factorisation of I into prime ideals, then $I^{-1} = \mathfrak{p}_1^{-1} \dots \mathfrak{p}_n^{-1}$.
   (b) Show that $II^{-1} = R$.

2. Let $\mathcal{O}_K$ be the ring of integers in a number field K and let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}_K$. Prove that there exists a unique prime number $p \in \mathbb{Z}$ such that $p \in \mathfrak{p}$.

3. Let θ be an algebraic integer, let f denote its minimal polynomial over $\mathbb{Q}$, and let $n = \deg f$. Let $K = \mathbb{Q}(\theta)$ and let $\sigma_1, \dots, \sigma_n$ be the embeddings of K into $\mathbb{C}$.

   Prove that
   $$\Delta\big(1, \theta, \dots, \theta^{n-1}\big) = (-1)^{\binom{n}{2}} \prod_{j=1}^{n} f'\big(\sigma_j(\theta)\big).$$
   In the special case of $f(x) = x^n + ax + b$ for fixed $a, b \in \mathbb{Q}$, show that
   $$\Delta\big(1, \theta, \dots, \theta^{n-1}\big) = (-1)^{\binom{n}{2}} \big(n^n b^{n-1} + a^n (1-n)^{n-1}\big).$$

4.
   (a) Prove that any number field of degree 2 is of the form $\mathbb{Q}(\sqrt{d})$ for some squarefree $d \in \mathbb{Z}$.
   (b) Prove that if $d_1 \neq d_2 \in \mathbb{Z}$ are squarefree, then the fields $\mathbb{Q}(\sqrt{d_1})$ and $\mathbb{Q}(\sqrt{d_2})$ are not isomorphic.
   (c) Fix d squarefree and let $K = \mathbb{Q}(\sqrt{d})$. Compute the discriminant of the ring of integers $\mathcal{O}_K$.

5. The following is an alternative construction of the ideal class group of a Dedekind ring R.
   (a) We say that two ideals I and J of R are *equivalent* if $aI = bJ$ for some nonzero $a, b \in R$. Prove that this is indeed an equivalence relation.
   (b) Suppose $I_1 \sim I_2$ and $J_1 \sim J_2$. Prove that $I_1 J_1 \sim I_2 J_2$.

      Use this to show that ideal multiplication defines an abelian group structure on the set $\widetilde{\mathrm{Cl}}(R)$ of equivalence classes of nonzero ideals of R.
   (c) Prove that $\widetilde{\mathrm{Cl}}(R)$ is isomorphic to $\mathrm{Cl}(R)$ as groups.

6. Let R be a Noetherian integral domain with fraction field K. Prove that $J \subseteq K$ is a fractional ideal of R if and only if it is a finitely-generated R-submodule of K.

7. Let K be a number field and $\beta \in K$. Let $m_\beta \colon K \to K$ denote the $\mathbb{Q}$-linear transformation given by $m_\beta(x) = \beta x$. Prove that
   $$|N(\beta)| = |\det(m_\beta)|.$$

8. Let $R = \mathbb{C}[X, Y]/(Y^2 - X^3)$. Is R a Dedekind domain?