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Trace Formulas of Hecke Operators and Their Vanishing

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Contents

Chapter 1

Introduction

Modular forms act as a useful tool in various areas such as number theory, complex analysis and mathematical physics. Their essential role in Andrew Wiles' solving Fermat's Last Theorem has drawn public attentions, and it has become pivotal in most contemporary research on number theory. Hecke operators are specific double coset operators acting on important subsets of modular forms called cusp forms, which will unveil interesting results about modular forms. My research will mainly focus on the traces of Hecke operators.

By the help of the Selberg trace formula Selberg [1956], M. Eichler constructed a trace formula Eichler [1973] to calculate the trace of a Hecke operator acting on cusp forms of a given weight, for a given congruence subgroup of the modular group. This is known as the Eichler-Selberg trace formula and it is our starting point for studies on the traces of Hecke operators.

The main target of the thesis is to determine trace zero levels of Hecke operators. J. Rouse in Rouse [2006] has offered an algorithm to find these trace zero levels for cusp forms of a given weight and for a given congruence subgroup of the modular group. In this thesis we will extend this algorithm to various circumstances.

This thesis is organised as follows. After introducing some necessary fundamentals of number theory in Chapter 2, in Chapter 3 we will give a brief tour of our main research objects: modular forms and Hecke operators. This part will end with a short description of the spaces of modular forms.

Then in Chapter 4, we will come to the main topic of the thesis: trace formulas of Hecke operators. After giving two different reformulations of the Eichler-Selberg trace formula, we will introduce some relevant trace formulas in the second half of this chapter.

In Chapter 5, we will first describe Rouse's algorithm in Section 5.2, and then extend it to different circumstances in the rest of the chapter using the trace formulas in Chapter 4.

Lastly, we will discuss some applications of traces of Hecke operators in Chapter 6.

Chapter 2

Fundamentals of Number Theory

In this Chapter, we will introduce some fundamental concepts and notions in number theory to provide the necessary background for readers.

2.1 Arithmetic Functions

In number theory, various arithmetic functions are introduced to display arithmetic properties of positive integers. Let $\mathbb{Z}^+$ denote all positive integers.

Definition 2.1 (Arithmetic Functions)**.** An arithmetic function is a function $f : \mathbb{Z}^+ \to \mathbb{C}$.

By convention, for any arithmetic function f, let $f(x) = 0$ if x is not an integer.

Definition 2.2 (Multiplicative)**.** A function $f : \mathbb{Z} \to \mathbb{C}$ is said to be multiplicative if $f(ab) = f(a)f(b)$ whenever $(a, b) = 1$. f is completely multiplicative if $f(ab) = f(a)f(b)$ for any $a, b \in \mathbb{Z}$.

Here are several important examples of arithmetic functions; for their properties, see [Ireland and Rosen, 2013, Section 2.2] or [Apostol, 2013, Chapter 2].

Definition 2.3 (Order of Prime Divisors)**.** For any prime p and any integer n, define $\mathrm{ord}_p n$ to be the largest integer l such that $p^l \mid n$.

Definition 2.4 (Euler ϕ-function)**.** The Euler's function $\phi(n)$ counts the number of integers between 1 and n relatively prime to n. It satisfies:

$$\phi(n) = n \sum_{p|n} \left(1 - \frac{1}{p}\right),$$

where p runs through all positive prime divisors of n.

Definition 2.5 (Möbius Function)**.** For $n \in \mathbb{Z}^+$, $\mu(n)$ is defined as:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is not square-free} \\ (-1)^\ell & \text{if } n = p_1 p_2 \dots p_\ell \end{cases}$$

where $p_1, \dots, p_\ell$ are distinct positive primes.

Remark 2.6. It is easy to see that the Möbius function is multiplicative. So is the Euler ϕ-function. See [Apostol, 2013, Theorem 2.5].

We have the following useful formula about the Möbius function.

Theorem 2.7 (Möbius Inversion theorem)**.** *For arithmetic functions f and g,*

$$f(n) = \sum_{d|n} g(d)$$

if and only if

$$g(n) = \sum_{d|n} f(d)\, \mu\left(\frac{n}{d}\right).$$

Proof. See [Apostol, 2013, Theorem 2.9] or [Ireland and Rosen, 2013, Theorem 2, Chapter 2]. ☐

Definition 2.8 (Legendre Symbol)**.** If $(n, m) = 1$, n is called a quadratic residue mod m if the congruence $x^2 \equiv n \mod m$ has a solution. Otherwise n is called a quadratic nonresidue mod m. Then for odd prime p, the Legendre symbol is defined to be:

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } n \text{ is a quadratic residue mod } p \\ -1 & \text{if } n \text{ is a quadratic nonresidue mod} p \\ 0 & \text{if } p \mid n. \end{cases}$$

It can be generalised to the Kronecker Symbol.

Definition 2.9 (Kronecker Symbol)**.** Let $\left(\frac{a}{1}\right) = 1$ and

$$\left(\frac{a}{2}\right) = \begin{cases} 0 & \text{if } a \text{ is even} \\ 1 & \text{if } a \equiv \pm 1 \mod 8 \\ -1 & \text{if } a \equiv \pm 3 \mod 8. \end{cases}$$

If $n > 2$ is an integer with prime factorisation $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ and $a \in \mathbb{Z}$, define the Kronecker symbol

$$\left(\frac{a}{n}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{e_i},$$

where $(\frac{a}{p_i})$ is the Legendre symbol when p_i is an odd prime number.

Given any odd prime p, the Legendre symbol is an example of Dirichlet characters modulo p.

Definition 2.10 (Dirichlet Character)**.** A function $\chi : \mathbb{Z} \to \mathbb{C}^*$ is a Dirichlet character modulo m if it satisfies the following three conditions:

(i) $\chi(n + m) = \chi(n)$ for all $n \in \mathbb{Z}$

(ii) $\chi(kn) = \chi(k)\chi(n)$ for all $k, n \in \mathbb{Z}$

(iii) $\chi(n) \neq 0$ if and only if $(n, m) = 1$

By definition, Dirichlet characters are completely multiplicative, and $\chi(n) = 0$ if n is not an integer by convention. Since Dirichlet characters are important in modular form theory, we then give some more definitions and properties about them.

Definition 2.11. The trivial Dirichlet character $\mathbb{1}$ modulo N is defined as

$$\mathbb{1}(n) = \begin{cases} 1 & \text{if } (N, n) = 1 \\ 0 & \text{if } (N, n) > 1. \end{cases}$$

For positive integers $d \mid N$ and Dirichlet characters χ modulo N and χ_d modulo d, if $\chi(n) = \chi_d(n)$ for all $n \in \mathbb{Z}$, then we say χ_d *lifts* to χ or χ_d is a restriction of χ.

The *conductor* $\mathfrak{f}$ of a Dirichlet character χ modulo N is the smallest integer such that there exists a Dirichlet character $\chi_{\mathfrak{f}}$ modulo $\mathfrak{f}$ that can lift to χ. A Dirichlet character χ modulo N is *primitive* if its conductor is N.

2.2 Quadratic Congruences

To solve a quadratic congruence $x^2 \equiv a \mod N$, we usually first solve the congruences $x^2 \equiv a \mod p^{\mathrm{ord}_p N}$ for each prime divisor p of N, and then combine the results by the Chinese remainder theorem. For each congruence $x^2 \equiv a \mod p^{\mathrm{ord}_p N}$, its solutions may possibly be lifted from solutions to $x^2 \equiv a \mod p$ by Hensel's lemma, and the number of solutions to the latter congruence can be represented by the Legendre symbol $(\frac{a}{p})$. In this section, we will state the theorems mentioned above.

Theorem 2.12 (Chinese Remainder Theorem)**.** *Suppose that $N = m_1 m_2 \dots m_n$ and $(m_i, m_j) = 1$ for $i \neq j$. Let $a_1, a_2, \dots, a_n$ be integers and consider the following system of congruences:*

$$x \equiv a_i \mod m_i \qquad \textit{for } i = 1, \dots, n.$$

The solutions to the system exist, and are unique up to adding a multiple of N.

Proof. A proof can be found in [Ireland and Rosen, 2013, Section 3.4]. □

Lemma 2.13 (Hensel's Lemma)**.** *Let $f \in \mathbb{Z}[x]$ and f' be its derivative. Consider $x, n, k \in \mathbb{Z}$ such that $0 \leq 2k < n$, $f(x) \equiv 0 \mod p^n, \mathrm{ord}_p(f'(x)) = k$. Then there exists $y \in \mathbb{Z}$ such that*

$$f(y) \equiv 0 \mod p^{n+1}, \ \mathrm{ord}_p(f'(y)) = k, \ \textit{and } y \equiv x \mod p^{n-k}.$$

Proof. This can be found in [Serre, 2012, Section 2.2]. □

In addition, one can use the quadratic reciprocity law to simplify the calculation of Legendre symbols.

Theorem 2.14 (Quadratic Reciprocity Law)**.** *Let p, q be distinct odd prime numbers. Then*

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)(-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

Proof. A proof can be found in [Serre, 2012, Section 3.3]. □

2.3 The Class Number

The target of this section is to define the class number $h(D)$ for binary quadratic forms of discriminant $D > 0$. This number appears in the trace formula of Hecke operators, and the formula for it will be given at the end as Corollary 2.27.

We start this section by introducing some terminology of integral quadratic forms in two variables:

$$f(x,y) = ax^2 + bxy + cy^2, \quad a,b,c \in \mathbb{Z}.$$

Definition 2.15 (Primitive)**.** A quadratic form $f(x,y) = ax^2+bxy+cy^2$ is *primitive* if a,b,c are relatively prime.

Definition 2.16 (Discriminant)**.** Define the *discriminant* of a quadratic form $f(x,y) = ax^2+bxy+cy^2$ to be $D = b^2-4ac$. The form is called *positive definite* (resp. *negative definite*) if $D < 0$ and $a > 0$ (resp. $a < 0$). The form is called *indefinite* if $D > 0$.

Definition 2.17 (Properly Equivalent)**.** Two quadratic forms are *properly equivalent* if there exist integers p,q,r,s such that $f(x,y) = g(px+qy, rx+sy)$ and $ps-qr = 1$. Since $\det\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) = 1$, we have that $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and proper equivalence is an equivalent relation (see [Cox, 2011, Section 2.A]).

Definition 2.18 (Reduced Form)**.** A primitive positive definite quadratic form $f(x,y) = ax^2 + bxy + cy^2$ satisfying:

(i) $|b| \leq a \leq c$

(ii) $b \geq 0$ when $|b| = a$ or $a = c$

is called a *reduced* form.

Now we have enough ingredients to define the class number $h(D)$ and offer a basic way to calculate it.

Definition 2.19 (Class Number)**.** For fixed $D < 0$, $h(D)$ is defined to be the number of proper equivalence classes of primitive positive definite quadratic forms of discriminant D. By convention $h(D) = 0$ if D is not an integer.

The following theorem provides a way to calculate it.

Theorem 2.20. *Every primitive positive definite quadratic form is properly equivalent to a unique reduced form. Therefore, $h(D)$ is finite and equal to the number of reduced forms of discriminant D.*

Proof. It can be found in [Cox, 2011, Section 2.A]. □

By this theorem, we can thus calculate the class number by listing and counting all the reduced forms of given discriminant D. By (i) in the definition of reduced forms, we have that $0 \leq b^2 \leq ac$, so $4ac = b^2 - D \implies -D/3 \geq ac \geq -D/4$. Consequently, in order to list all reduced forms of discriminant $D < 0$, we can first list the finite set of quadratic forms $ax^2 + bxy + cy^2$ satisfying $-D/3 \geq ac \geq -D/4$ and $|b| \leq a \leq c$, and then screen out those that do not satisfy the definition of reduced forms. It is possible to calculate the class number in this way, but the workload is growing tremendous when $|D|$ increases. We will therefore give a closed-form expression of $h(D)$.

To reach this, the bijection between classes of primitive positive definite quadratic forms of discriminant D and the ideal class group of the imaginary quadratic order $\mathcal{O}$ of discriminant D is used. Terminology and proof details can be found in [Cox, 2011, Section 7.A-7.C]. Before we give the formula of $h(D)$, we first introduce some ideas from algebraic number theory.

Definition 2.21 (Imaginary Quadratic Field)**.** An *imaginary quadratic field* K is an extension of $\mathbb{Q}$ of the form $K = \mathbb{Q}(\sqrt{m})$ with square-free $m < 0$. The discriminant of K is

$$d_k = \begin{cases} m & \text{if } m \equiv 1 \mod 4 \\ 4m & \text{otherwise.} \end{cases}$$

Definition 2.22 (Order)**.** An *order* in an imaginary quadratic field K is a subring $\mathcal{O}$ of K (containing 1) such that

(a) $\mathcal{O}$ is a finitely generated $\mathbb{Z}$-module;

(b) $\mathcal{O}$ contains a $\mathbb{Q}$-basis of K.

An example of an order is the *ring of integers* $\mathcal{O}_K$ of K defined by

$$\mathcal{O}_K = \mathbb{Z}\left[\frac{d_K + \sqrt{d_K}}{2}\right].$$

In fact, we have the following lemma:

Lemma 2.23. *Every order of K is of the form $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ for some $f \in \mathbb{Z}^+$.*

Proof. This is [Cox, 2011, Lemma 7.2]. □

Then the *discriminant* of an order $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ is defined to be $d = f^2 d_K$.

Proposition 2.24. *Let μ_n be the group of n^{th} roots of unity. Given an order $\mathcal{O}$, its group of units is*

$$\mathcal{O}^* = \begin{cases} \mu_4 & \text{if } \mathcal{O} \text{ is an order of discriminant } -4 \\ \mu_6 & \text{if } \mathcal{O} \text{ is an order of discriminant } -3 \\ \{\pm 1\} & \text{otherwise.} \end{cases}$$

Proof. A proof can be seen in [Knightly and Li, 2006, Lemma 26.8]. □

Now we finally have the closed-form expression of $h(D)$.

Theorem 2.25. *For any $\mathcal{O}_K$ the ring of integers of K, which is also an order of discriminant d_K, we have*

$$h(d_K) = -\frac{|\mathcal{O}_K^*|}{2|d_K|} \sum_{n=1}^{|d_K|-1} \left(\frac{d_K}{n}\right) n,$$

where $(\frac{d_K}{n})$ is the Kronecker symbol.

Proof. This formula can be proved by analytic methods, such as [Borevich and Shafarevich, 1986, Section 5.4] or [Zagier, 1981, § 9]. An algebraic proof can be found in Orde [1978]. □

Theorem 2.26. *Let $D \equiv 0, 1 \mod 4$ be negative and $m \in \mathbb{Z}^+$. Then*

$$h(m^2 D) = \frac{h(D)m}{[\mathcal{O}^* : \mathcal{O}'^*]} \prod_{p|m} \left(1 - \left(\frac{D}{p}\right)\frac{1}{p}\right),$$

where $\mathcal{O}$ and $\mathcal{O}'$ are the orders of discriminant D and $m^2 D$ respectively, and $\mathcal{O}'$ has index m in $\mathcal{O}$.

Proof. It can be found in [Cox, 2011, Section 7.D]. □

Combining Theorem 2.25 and Theorem 2.26, we can derive a method to calculate $h(D)$ where $D < 0$ and $D \equiv 0, 1 \mod 4$. First, we can decompose $D = m^2 d$ where $m \in \mathbb{Z}^+$ and d is square-free. If $d \equiv 1 \mod 4$, then we can calculate $h(d)$ by Theorem 2.25, and then calculate $h(m^2 d)$ by Theorem 2.26. If $d \equiv 2, 3 \mod 4$, then since $t^2 \equiv 0, 1 \mod 4$ for any $t \in \mathbb{Z}$, we have that $m^2 \equiv 0 \mod 4$. Therefore, we can decompose $D = (m/2)^2(4d)$, and then calculate $h(4d)$ by Theorem 2.25, and calculate $h((m/2)^2(4d))$ by Theorem 2.26. This can be stated as:

Corollary 2.27 (Closed-Form Formula of $h(D)$)**.** *Let $D \equiv 0, 1 \mod 4$ be negative. Decompose $D = \tilde{m}^2\tilde{d}$ where $\tilde{m} \in \mathbb{Z}^+$ and $\tilde{d}$ is square-free. If $\tilde{d} \equiv 1 \mod 4$, then $m = \tilde{m}$, $d = \tilde{d}$. Otherwise, $m = \tilde{m}/2$, $d = 4\tilde{d}$. Then*

$$h(D) = -\frac{m|\mathcal{O}^*|}{2|d|[\mathcal{O}^* : \mathcal{O}'^*]}\left[\sum_{n=1}^{|d|-1}\left(\frac{d}{n}\right)n\right]\prod_{p|m}\left(1-\left(\frac{d}{p}\right)\frac{1}{p}\right),$$

where $\mathcal{O}$ and $\mathcal{O}'$ are the orders of discriminant d and D respectively, and $(\frac{d}{n})$ is the Kronecker symbol. □

Chapter 3

Modular Forms and Hecke Operators

In this chapter, we briefly introduce our main research objects: modular forms and Hecke operators. After stating the definition of Hecke operators, we define newforms and oldforms, and describe the structure of $\mathcal{M}_k(\Gamma)$. The main reference of this chapter is Diamond and Shurman [2005].

3.1 Modular Forms and Cusp Forms

To define modular forms with respect to a congruence group, we first give some background concepts. Let $\mathcal{H}$ be the upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.

Definition 3.1 (Congruence Subgroup)**.** The *modular group* is the group $\mathrm{SL}_2(\mathbb{Z})$. Let N be a positive integer. The *principal congruence subgroup* of level N is

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod N \right\}.$$

Moreover, Γ is a *congruence subgroup* of level N if $\Gamma(N) \subseteq \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$.

Each matrix in the modular group can be regarded as an automorphism of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as the following:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau) = \frac{a\tau + b}{c\tau + d}, \quad \tau \in \hat{\mathbb{C}}.$$

We give two important examples of congruence subgroups.

Definition 3.2 ($\Gamma_0(N)$ and $\Gamma_1(N)$)**.** Let N be a positive integer. We define

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \mod N \right\};$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \mod N \right\}.$$

We can see that

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z}).$$

Definition 3.3 (Factor of Automorphy)**.** For any matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$, the factor of automorphy $j(\gamma, \tau) \in \mathbb{C}$ is

$$j(\gamma, \tau) = c\tau + d.$$

Let $\mathrm{GL}_2^+(\mathbb{Q})$ denote the group of 2×2 matrices with positive determinant and rational entries.

Definition 3.4 (Weight-k Operator)**.** For $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ and integer k, we define the weight-k operator $[\gamma]_k$ acting on functions $f : \mathcal{H} \to \mathbb{C}$ by

$$(f[\gamma]_k)(\tau) = (\det\gamma)^{k-1} j(\gamma, \tau)^{-k} f(\gamma(\tau)), \tau \in \mathcal{H}.$$

Definition 3.5 (Weakly Modular of Weight k)**.** A meromorphic function f on $\mathcal{H}$ is weakly modular of weight k with respect to a congruence subgroup Γ if $f[\gamma]_k = f$ for any $\gamma \in \Gamma$.

Each congruence subgroup Γ contains a translation matrix $\gamma = \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ for some minimal $h \in \mathbb{Z}^+$ by the definition of congruence subgroup. Thus, every weakly modular function $f : \mathcal{H} \to \mathbb{C}$ with respect to Γ is $h\mathbb{Z}$-periodic, so the function $g : D' \to \mathbb{C}$ from the punctured unit disk to the complex plane is well defined as $g(q) = f(\tau)$ where $q = e^{2\pi i \tau / h}$.

Definition 3.6 (Holomorphic at ∞)**.** For a holomorphic weakly modular function $f : \mathcal{H} \to \mathbb{C}$ with respect to Γ, it is holomorphic at ∞ if the function $g : D' \to \mathbb{C}$ defined above can be extended to $q = 0$ holomorphically. Furthermore, such f has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau / h}.$$

Now we can define modular forms and cusp forms.

Definition 3.7 (Modular Forms)**.** Let Γ be a congruence subgroup of the modular group and let k be an integer. A modular form of weight k with respect to Γ is a function $f : \mathcal{H} \to \mathbb{C}$ such that

1. f is holomorphic on $\mathcal{H}$,
2. f is weakly modular of weight k with respect to Γ,
3. $f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Let $\mathcal{M}_k(\Gamma)$ denote the set of modular forms of weight k with respect to Γ.

Definition 3.8 (Cusp Forms)**.** A modular form f of weight k with respect to Γ is a cusp form if the constant term $a_0 = 0$ in the Fourier expansion of $f[\gamma]_k$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Let $\mathcal{S}_k(\Gamma)$ denote the set of cusp forms of weight k with respect to Γ.

We can check that $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$ both form vector spaces over $\mathbb{C}$.

The following subspaces of $\mathcal{M}_k(\Gamma_1(N))$ involving the Dirichlet characters are useful.

Definition 3.9 (χ-eigenspace of $\mathcal{M}_k(\Gamma_1(N))$)**.** For each Dirichlet character χ modulo N, we can define

$$\mathcal{M}_k(N,\chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)) : f[\gamma]_k = \chi(d_\gamma)f \text{ for any } \gamma \in \Gamma_0(N)\}$$

where d_γ is the lower right entry of γ.

Note that $\mathcal{M}_k(N,\mathbb{1}) = \mathcal{M}_k(\Gamma_0(N))$ and $\mathcal{M}_k(N,\chi) = \{0\}$ unless $\chi(-1) = (-1)^k$. [Diamond and Shurman, 2005, Section 4.3]

Here are some examples of modular forms.

Example 3.10. *The reference for these examples is [Diamond and Shurman, 2005, Section 1.1-1.2].*

(a) The Eisenstein series of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$ is defined as

$$G_k(\tau) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{(c\tau+d)^k}, \quad \tau \in \mathcal{H}.$$

This sum is absolutely convergent and converges uniformly on compact subsets of $\mathcal{H}$, so it is holomorphic on $\mathcal{H}$. It is a modular form of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$.

(b) Consider $G_{2,N} : \mathcal{H} \to \mathbb{C}$ defined by

$$G_{2,N}(\tau) = G_2(\tau) - NG_2(N\tau).$$

This is a modular form of weight 2 with respect to $\Gamma_0(N)$.

(c) The discriminant function $\Delta : \mathcal{H} \to \mathbb{C}$ defined by

$$\Delta(\tau) = (60G_4(\tau))^3 - 27(140G_6(\tau))^2$$

is a cusp form of weight 12 with respect to $\mathrm{SL}_2(\mathbb{Z})$.

On the other hand, the function

$$\eta^{24}(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24},\ q = e^{2\pi i\tau}$$

is also a cusp form of weight 12 with respect to $\mathrm{SL}_2(\mathbb{Z})$.

Since $\mathcal{S}_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is 1-dimensional, either Δ or η^{24} spans this space. In fact,

$$\Delta = (2\pi)^{12}\eta^{24}.$$

3.2 Hecke Operators

In this section we will give the definition of Hecke operators. Since Hecke operators are double coset operators, we will first introduce them.

Definition 3.11 (Double Coset Operators)**.** For congruence subgroups Γ and Γ' of $\mathrm{SL}_2(\mathbb{Z})$ and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, a double coset operator is a weight-k $\Gamma\alpha\Gamma'$ operator taking modular forms $f \in \mathcal{M}_k(\Gamma)$ to:

$$f[\Gamma\alpha\Gamma']_k = \sum_j f[\beta_j]_k$$

where $\{\beta_j\}$ are orbit representatives which means that $\Gamma\alpha\Gamma' = \cup_j \Gamma\beta_j$ is a disjoint union.

To show that the double coset operator is well defined, we need to show the existence of orbit representatives and its independence of how the orbit representatives are chosen. These are proved in [Diamond and Shurman, 2005, Section 5.1].

We can also notice that $[\Gamma\alpha\Gamma']_k$ takes $\mathcal{M}_k(\Gamma)$ to $\mathcal{M}_k(\Gamma')$, and takes $\mathcal{S}_k(\Gamma)$ to $\mathcal{S}_k(\Gamma')$. See [Diamond and Shurman, 2005, Section 5.1] for detail.

Hecke operators are specific double coset operators with $\Gamma = \Gamma' = \Gamma_1(N)$ for some positive integer N. There are two types of Hecke operators, and we will focus on the second type starting from Chapter 4.

Definition 3.12 (Hecke Operators $\langle d\rangle$ and T_p)**.** The first type of Hecke operators $\langle d\rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^*$ is defined to be

$$\langle d\rangle = [\Gamma_1(N)\begin{bmatrix} a & b \\ c & d\end{bmatrix}\Gamma_1(N)]_k, \quad \begin{bmatrix} a & b \\ c & d\end{bmatrix} \in \Gamma_0(N).$$

This is well defined, i.e. the action of the operators is independent of the selection of a, b, c ([Diamond and Shurman, 2005, Section 5.2]). The second type of Hecke operators T_p for prime p is

$$T_p = [\Gamma_1(N)\begin{bmatrix} 1 & 0 \\ 0 & p\end{bmatrix}\Gamma_1(N)]_k.$$

We give a specific expression of both operators ([Diamond and Shurman, 2005, Section 5.2]).

Proposition 3.13. *For any $f \in \mathcal{M}_k(\Gamma_1(N))$,*

$$\langle d\rangle f = f[\alpha]_k \text{ for any } \alpha = \begin{bmatrix} a & b \\ c & \delta\end{bmatrix} \in \Gamma_0(N), \delta \equiv d \mod N.$$

and

$$T_p f = \begin{cases} \displaystyle\sum_{j=0}^{p-1} f\left[\begin{bmatrix} 1 & j \\ 0 & p\end{bmatrix}\right]_k & \text{if } p \mid N, \\ \displaystyle\sum_{j=0}^{p-1} f\left[\begin{bmatrix} 1 & j \\ 0 & p\end{bmatrix}\right]_k + f\left[\begin{bmatrix} m & n \\ N & p\end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & 1\end{bmatrix}\right]_k & \text{if } p \nmid N, \text{ where } mp - nN = 1. \end{cases}$$

□

In addition, the Hecke operators commute ([Diamond and Shurman, 2005, Section 5.2]). Moreover, Hecke operators $\langle d\rangle$ and T_p can be extended to $\langle n\rangle$ and T_n for $n \in \mathbb{Z}^+$.

Definition 3.14 (Hecke Operators $\langle n\rangle$ and T_n)**.** For any integer n with prime factorisation $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$, we can define Hecke operators acting on $\mathcal{S}_k(\Gamma_1(N))$:

$$\langle n\rangle = \begin{cases} 0 & \text{if } (n, N) \neq 1 \\ \langle n\rangle & \text{if } (n, N) = 1, \end{cases}$$

and

$$T_n = \begin{cases} 1 & \text{if } n = 1 \\ \prod_j T_{{p_j}^{e_j}} & \text{if } n \neq 1, \end{cases}$$

where T_{p^e} for prime p and $2 \le e \in \mathbb{Z}$ is defined inductively

$$T_{p^e} = T_p T_{p^{e-1}} - p^{k-1}\langle p\rangle T_{p^{e-2}}.$$

These are well defined. See [Diamond and Shurman, 2005, Section 5.3] for detail.

3.3 Atkin-Lehner theory

In this section we introduce newforms and oldforms. We start our topic by considering taking modular forms from lower levels $M \mid N$ up to level N.

We can notice that for $M \mid N$ we have $\mathcal{S}_k(\Gamma_1(M)) \subset \mathcal{S}_k(\Gamma_1(N))$, so the inclusion is a trivial way to move modular forms in $\mathcal{S}_k(\Gamma_1(M))$ to $\mathcal{S}_k(\Gamma_1(N))$. There is another way to embed $\mathcal{S}_k(\Gamma_1(M))$ into $\mathcal{S}_k(\Gamma_1(N))$. For $d \mid (N/M)$, let $\alpha_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so $[\alpha_d]_k$ takes $\mathcal{S}_k(\Gamma_1(M))$ to $\mathcal{S}_k(\Gamma_1(N))$. See [Diamond and Shurman, 2005, Section 5.6]. Now we can define the subspace of oldforms and newforms by distinguishing the part of $\mathcal{S}_k(\Gamma_1(N))$ coming from lower level or not.

Definition 3.15 (Subspace of Oldforms)**.** For every positive divisor d of N, let i_d be

$$i_d : \mathcal{S}_k(\Gamma_1(N/d)) \times \mathcal{S}_k(\Gamma_1(N/d)) \to \mathcal{S}_k(\Gamma_1(N))$$

defined by

$$(f, g) \mapsto f + \bar{i}_d(g) := f + g[\alpha_d]_k.$$

The subspace of oldforms at level N is

$$\mathcal{S}_k(\Gamma_1(N))^{\text{old}} = \sum_{\substack{\text{prime} \\ p|N}} i_p((\mathcal{S}_k(\Gamma_1(N/p)))^2).$$

We want to define the orthogonal complement of the subspace of oldforms to be the subspace of newforms, so we first need to make the space of cusp forms of given weight k be an inner product space.

Definition 3.16 (Petersson Inner Product)**.** Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The hyperbolic measure on $\mathcal{H}$ is

$$d\mu(\tau) = \frac{dx\,dy}{y^2}, \quad \tau = x + iy \in \mathcal{H}.$$

Then the Petersson inner product $\langle , \rangle_\Gamma : \mathcal{S}_k(\Gamma) \times \mathcal{S}_k(\Gamma) \to \mathbb{C}$ is given by

$$\langle f, g \rangle_\Gamma = \frac{1}{V_\Gamma} \int_{X(\Gamma)} f(\tau)\overline{g(\tau)}(\mathrm{Im}(\tau))^k d\mu(\tau)$$

where $\mathrm{Im}(\tau)$ means the imaginary part of τ, $X(\Gamma)$ is the modular curve with respect to the congruence subgroup Γ, $V_\Gamma = \int_{X(\Gamma)} d\mu(\tau)$.

The modular curve $X(\Gamma) = \{\Gamma s \mid s \in \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}\}$. See [Diamond and Shurman, 2005, Chapter 2] for its further properties and [Diamond and Shurman, 2005, Section 5.4] for how to integrate over $X(\Gamma)$. The Petersson inner product is a well defined inner product. Again see [Diamond and Shurman, 2005, Section 5.4] for detail.

Now we can define the subspace of newforms.

Definition 3.17 (Subspace of Newforms)**.** The subspace of newforms at level N is the orthogonal complement of the space of oldforms with respect to the Petersson inner product

$$\mathcal{S}_k(\Gamma_1(N))^{\text{new}} = (\mathcal{S}_k(\Gamma_1(N))^{\text{old}})^\perp.$$

The Hecke operators respect this decomposition of $\mathcal{S}_k(\Gamma_1(N))$.

Proposition 3.18. *The subspaces $\mathcal{S}_k(\Gamma_1(N))^{\text{old}}$ and $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ are stable under the Hecke operators T_n and $\langle n \rangle$ for all $n \in \mathbb{Z}^+$. Furthermore, these two subspaces have orthogonal bases of eigenforms for the Hecke operators at levels coprime to N, $\{T_n, \langle n \rangle : (n, N) = 1\}$.*

Proof. See [Diamond and Shurman, 2005, Section 5.4]. □

The latter conclusion can be strengthened for $\mathcal{S}_k(\Gamma_1(N))^{\mathrm{new}}$.

Theorem 3.19. *Consider the subspace of newforms $\mathcal{S}_k(\Gamma_1(N))^{\mathrm{new}}$:*

(a) Let $f \in \mathcal{S}_k(\Gamma_1(N))^{\mathrm{new}}$ be an eigenform for the Hecke operators T_n and $\langle n \rangle$ with $(n, N) = 1$. Then f is an eigenform for T_n and $\langle n \rangle$ for all $n \in \mathbb{Z}^+$. If there is another $\hat{f}$ that satisfies the same condition and has the same T_n-eigenvalues as f, then $\hat{f} = cf$ for some constant c.

(b) The set of these eigenforms with Fourier coefficient $a_1 = 1$ form an orthogonal basis of $\mathcal{S}_k(\Gamma_1(N))^{\mathrm{new}}$, and each of them has T_n-eigenvalues equal to its Fourier coefficient $a_n(f)$, i.e. $T_n f = a_n(f) f$ for all $n \in \mathbb{Z}^+$.

Proof. See [Diamond and Shurman, 2005, Section 5.8]. □

3.4 Structure of the Space of Modular Forms

For any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, we have the following decomposition:

$$\mathcal{M}_k(\Gamma) = \mathcal{S}_k(\Gamma) \oplus \mathcal{E}_k(\Gamma),$$

where $\mathcal{E}_k(\Gamma)$ is the space of Eisenstein series for Γ. See [Diamond and Shurman, 2005, Chapter 4] for the definition and properties of Eisenstein series with respect to a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

When $\Gamma = \Gamma_1(N)$ given $N \in \mathbb{Z}^+$, we can decompose $\mathcal{M}_k(\Gamma_1(N))$ into eigenspaces of Hecke operators.

Consider the first type of Hecke operators $\langle n \rangle$. We have that $\mathcal{M}_k(N, \chi)$ for any Dirichlet character are precisely the eigenspaces of $\{\langle n \rangle : n \in \mathbb{Z}^+\}$

$$\mathcal{M}_k(N, \chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)) : \langle n \rangle f = \chi(n) f,\ \forall n \in \mathbb{Z}^+\},$$

and thus there exists a decomposition

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{M}_k(N, \chi).$$

The same results apply for $\mathcal{S}_k(N,\chi)$ and $\mathcal{E}_k(N,\chi)$. See [Diamond and Shurman, 2005, Section 4.3, 5.2] for detail.

Consider both two types of Hecke operators $\{T_n, \langle n\rangle : n \in \mathbb{Z}^+\}$. For $\mathcal{S}_k(\Gamma_1(N))$, its decomposition into the subspaces of newforms and oldforms

$$\mathcal{S}_k(\Gamma_1(N)) = \mathcal{S}_k(\Gamma_1(N))^{\text{new}} \oplus \sum_{\substack{\text{prime} \\ p|N}} i_p((\mathcal{S}_k(\Gamma_1(N/p)))^2).$$

can be more precise: each of the summands can be decomposed into 1-dimensional eigenspaces for Hecke operators $\{T_n, \langle n\rangle : n \in \mathbb{Z}^+\}$ by Theorem 3.19. Then we can see that $\mathcal{S}_k(\Gamma_1(N))$ has a basis [Diamond and Shurman, 2005, Section 5.8]

$$\mathcal{B}_k(N) = \{f(n\tau) : f \in \mathcal{S}_k(\Gamma_1(M))^{\text{new}} \text{ and } nM \mid N\}$$

where $f \in \mathcal{S}_k(\Gamma_1(M))^{\text{new}}$ is any eigenform for the Hecke operators $\{T_n, \langle n\rangle : n \in \mathbb{Z}^+\}$ with Fourier coefficient $a_1 = 1$.

Additionally, there are similar decompositions of the Eisenstein series with respect to the Hecke operators [Diamond and Shurman, 2005, Section 5.2].

Chapter 4

Trace Formulas of Hecke Operators

In this chapter, we give trace formulas of Hecke operators acting on different subspaces of modular forms. In the first half 4.1 - 4.2 we will focus on the Eichler-Selberg trace formula, which considers Hecke operators T_n acting on $\mathcal{S}_k(N,\chi)$ for some $k,n,N \in \mathbb{Z}^+$ and a Dirichlet character χ modulo N, especially when $\chi = \mathbb{1}$ and thus $\mathcal{S}_k(N,\chi) = \mathcal{S}_k(\Gamma_0(N))$. We will give two versions of the Eichler-Selberg trace formula which were derived independently.

In the latter half 4.3 - 4.4, we will state more trace formulas for other subspaces of $\mathcal{M}_k(N,\chi)$, including $\mathcal{S}_k(\Gamma_0(N))^{\text{new}}$, W_l acting on $\mathcal{S}_k(\Gamma_0(N))$ and $\mathcal{S}_k(\Gamma_1(N))$, where W_l is the Atkin-Lehner operator to be introduced in Section **??**.

4.1 Preliminaries

We begin by introducing some functions appearing in the trace formulas repeatedly.

Definition 4.1 (Kronecker delta function)**.** For two objects i and j, define

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.2. Denote the index of $\Gamma_0(N)$ in the modular group $\mathrm{SL}_2(\mathbb{Z})$:

$$\phi_1(N) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1 + \frac{1}{p})$$

where p ranges over all positive prime divisors of N.

Definition 4.3. Let $D < 0$ be an integer, and let $\mathcal{O}$ be an order of determinant D. Denote

$$h_0(D) = \frac{2h(D)}{|\mathcal{O}^*|}$$

where $|\mathcal{O}^*|$ is the number of units in $\mathcal{O}$.

4.2 Two Versions of Eichler-Selberg Trace Formula

M. Eichler (Eichler [1973, 1957]) applied the Selberg trace formula (Selberg [1956]) to the trace of Hecke operators acting on cusp forms, and obtained the Eichler-Selberg trace formula. The Eichler-Selberg trace formula can be used to calculate the trace of a Hecke operator T_n acting on the vector space of cusp forms $\mathcal{S}_k(N, \chi)$ of a given weight k, for the congruence subgroup $\Gamma_1(N)$ of the modular group and a Dirichlet character χ modulo N. However, his original paper only justified his formula when N is square-free and $(N, n) = 1$.

Later, H. Hijikata (Hijikata [1974]) derived a trace formula of Hecke operators T_n acting on $\mathcal{S}_k(N, \chi)$, independent of Eichler's original conclusion. This seemingly different trace formula can also be applied to the case when N is not square-free. After that, Eichler's original work was also adapted to the case when N is not square-free and $(N, n) \neq 1$. See Knightly and Li [2006] or Cohen [1977], Oesterlé [1977] for detail.

We first introduce Eichler's version of the Eichler-Selberg trace formula after clarifying some notations.

Definition 4.4. For $s \in \mathbb{Z}$, $N, n, f \in \mathbb{Z}^+$ and χ a Dirichlet character modulo N, define

$$B(N, \chi, f, s, n) = \frac{\phi_1(N)}{\phi_1(N/f)} \sum_{x \in S_N(f,s,n)} \chi(x),$$

where $S_N(f, s, n) = \{\alpha \in (\mathbb{Z}/N\mathbb{Z})^\times : \alpha^2 - s\alpha + n \equiv 0 \mod Nf\}$.

Definition 4.5. For $N, a, d \in \mathbb{Z}^+$ and χ a Dirichlet character modulo N, define

$$\Phi(N, \chi, a, d) = \sum_{\substack{N=rs \\ (r,s)|(N/\mathfrak{f}, a-d)}} \phi((r, s))\chi(\alpha),$$

where r, s are positive divisors of N, α is an integer modulo $N/(r, s)$ such that $\alpha \equiv a \mod r$ and $\alpha \equiv d \mod s$, $\mathfrak{f}$ is the conductor of χ, and ϕ is the Euler's function.

Now we can give the Eichler-Selberg trace formula.

Theorem 4.6. *Let $N, n \geq 1$ and $k \geq 2$ be integers, and χ a Dirichlet character modulo N with $\chi(-1) = (-1)^k$. Then the trace of the Hecke operator T_n acting on $\mathcal{S}_k(N, \chi)$ is:*

$$\mathrm{Tr}(T_n, \mathcal{S}_k(N, \chi)) = -\frac{1}{2}\sum_{s^2<4n} p_{k-2}(s, n)\sum_f h_0\left(\frac{s^2-4n}{f^2}\right) B(N, \chi, (N, f), s, n) \tag{4.2.1a}$$

$$+\frac{\phi_1(N)}{12}(k-1)n^{k/2-1}\chi(\sqrt{n}) \tag{4.2.1b}$$

$$-\frac{1}{2}\sum_{a|n} \min(a, n/a)^{k-1}\Phi(N, \chi, a, n/a) \tag{4.2.1c}$$

$$+\delta_{k,2}\delta_{\chi,\mathbb{1}}\sum_{d|n,\ (N,d)=1} \frac{n}{d}, \tag{4.2.1d}$$

where d ranges over positive divisors of N, δ is the Kronecker delta function, h_0 is defined in Section 4.1 and p_{k-1} is defined as the coefficients of the following expansion:

$$\frac{1}{1 - sx + nx^2} = \sum_{w\geq 0} p_w(s, n)x^w.$$

Proof. See Popa [2014, 2017], Cohen [1977], Oesterlé [1977] or Knightly and Li [2006]. □

Since $h_0(x) = 0$ if x is not an integer by convention, there are finitely many nonzero summands in the sum over f. The terms 4.2.1a with $s^2 - 4n < 0$ are called the elliptic terms, and 4.2.1b is called the identity term. As for the terms 4.2.1c, it is called the unipotent term when $a = \sqrt{n}$, and the hyperbolic terms when $a \neq \sqrt{n}$.

Next we introduce Hijikata's version of trace formula. We again first define some notations.

Definition 4.7. For fixed $n \in \mathbb{Z}^+$ and $s \in \mathbb{Z}$ such that $s^2 - 4n < 0$ or $s^2 - 4n$ is a square. Let t_0 be the largest positive integer such that $t_0 \mid (s^2 - 4n)$. Then define

$$t(s,n) = \begin{cases} t_0 & \text{if } (s^2-4n)/t_0^2 \equiv 1 \mod 4 \\ t_0/2 & \text{if } (s^2-4n)/t_0^2 \equiv 2,3 \mod 4. \end{cases}$$

Definition 4.8. For $k, n \in \mathbb{Z}^+$ and $s \in \mathbb{Z}$ such that $s^2 - 4n < 0$ or $s^2 - 4n$ is a square, let $x_1, x_2 \in \mathbb{C}$ be the two roots of $\Psi(x) := x^2 - sx + n = 0$. Then define

$$a(s,k,n) = \begin{cases} \operatorname{sgn}(x)^k \frac{\min(|x_1|,|x_2|)^{k-1}}{|x-y|} & \text{if } s^2-4n > 0 \\ \frac{x_1^{k-1} - x_2^{k-1}}{2(x_1-x_2)} & \text{if } s^2-4n < 0 \end{cases}$$

Definition 4.9. For $n \in \mathbb{Z}^+$, $s \in \mathbb{Z}$ such that $s^2 - 4n < 0$ or $s^2 - 4n$ is a positive square, and $f \mid t(s,n)$, define

$$b(s,f,n) = \begin{cases} \frac{1}{2}\phi\left(\frac{\sqrt{s^2-4n}}{f}\right) & \text{if } s^2-4n > 0 \\ h_0\left(\frac{s^2-4n}{f^2}\right) & \text{if } s^2-4n < 0 \end{cases}$$

The next definition is essential in the next chapter.

Definition 4.10. Suppose $N, n > 1$ is fixed such that $(N,n) = 1$, p is a prime, $s \in \mathbb{Z}$ such that $s^2 - 4n$ is a square or is negative, $f \mid t(s,n)$, and a Dirichlet character χ modulo N. Let $v = \mathrm{ord}_p(N)$ and $b = \mathrm{ord}_p(f)$. Let $\chi = \prod_{p|N} \chi_p$ where p ranges over positive prime divisors of N and χ_q is a Dirichlet character modulo $q^{\mathrm{ord}_q N}$. Then let $\Psi(x) = x^2 - sx + n$

$$\hat{A}(s,f,N,n,p) = \{x \in \mathbb{Z}/p^{v+b}\mathbb{Z} : \Psi(x) \equiv 0 \mod p^{v+2b},\ \ 2x \equiv s \mod p^b\}$$
$$\hat{B}(s,f,N,n,p) = \{x \in \mathbb{Z}/p^{v+b}\mathbb{Z} : \Psi(x) \equiv 0 \mod p^{v+2b+1},\ \ 2x \equiv s \mod p^b\}$$

Also, we can let $\bar{A}(s,f,N,n,p)$ and $\bar{B}(s,f,N,n,p)$ be the sets of representatives of $\hat{A}(s,f,N,n,p)$ and $\hat{B}(s,f,N,n,p)$ respectively, and let $\bar{B}'(s,f,N,n,p) = \{s - x : x \in \bar{B}(s,f,N,n,p)\}$. Now we can define

$$c(s,f,N,n,p,\chi) = \begin{cases} \sum_x \chi_p(x) & \text{if } (s^2-4n)/f^2 \not\equiv 0 \mod p \\ \sum_x \chi_p(x) + \sum_y \chi_p(x) & \text{if } (s^2-4n)/f^2 \equiv 0 \mod p \end{cases}$$

where x ranges over $\bar{A}(s,f,N,n,p)$ and y ranges over $\bar{B}'(s,f,N,n,p)$.

Furthermore, define

$$c(s,f,N,n,\chi) = \prod_{p|N} c(s,f,N,n,p,\chi)$$

where p runs over positive prime divisors of N.

The solution sets $\hat{A}(s,f,N,n,p)$ and $\hat{B}(s,f,N,n,p)$ have been calculated explicitly in [Hijikata et al., 1989, Lemma 2.5]. By the definition, $c(s,f,N,n,\chi)$ is a multiplicative arithmetic function with respect to N, i.e. for any $N = N_1N_2$ with $(N_1,N_2) = 1$, we have $c(s,f,N,n,\chi) = c(s,f,N_1,n,\chi)\, c(s,f,N_2,n,\chi)$

Next we can give Hijikata's version of the Eichler-Selberg trace formula.

Theorem 4.11. *Let $N,n \geq 1$ and $k \geq 2$ be integers, and χ a Dirichlet character modulo N with $\chi(-1) = (-1)^k$. Then for $(N,n) = 1$, the trace of the Hecke operator T_n acting on $\mathcal{S}_k(N,\chi)$ is:*

$$\begin{aligned}\mathrm{Tr}(T_n, \mathcal{S}_k(N,\chi)) = &- \sum_s a(s,k,n) \sum_{f|t(s,n)} b(s,f,n)c(s,f,N,n,\chi)\\ &+ \frac{\phi_1(N)}{12}(k-1)n^{k/2-1}\chi(\sqrt{n})\\ &- n^{k/2-1}\chi(\sqrt{n})\frac{\sqrt{n}}{2}\prod_{\ell}\mathrm{par}(\ell)\\ &+ \delta_{k,2}\delta_{\chi,\mathbb{1}}\sum_{d|n}\frac{n}{d},\end{aligned}$$

where s runs over integers such that $s^2 - 4n$ is a positive square or is negative, d ranges over positive divisors of n, δ is the Kronecker delta function, h_0 is the class number defined in Section 4.1, ℓ ranges over positive prime divisors of N and $\mathrm{par}(\ell)$ *is defined as*

$$\mathrm{par}(\ell) = \begin{cases} 2\ell^{\nu-e} & \text{if } e \geq \rho+1\\ \ell^{\rho}+\ell^{\rho+1} & \text{if } e \leq \rho \text{ and } \nu \text{ is even}\\ 2\ell^{\rho} & \text{if } e \leq \rho \text{ and } \nu \text{ is odd}\end{cases}$$

where $\nu = \mathrm{ord}_\ell N$, $\rho = \lfloor \frac{\nu}{2} \rfloor$, $\mathfrak{f}$ is the conductor of χ_ℓ and then $e = \mathrm{ord}_\ell\, \mathfrak{f}$.

Proof. This was proved in a more general setting in Hijikata [1974]. A proof for this more exact setting of the trace formula can be found in [Hijikata et al., 1990, Theorem 2.2]. □

4.3 Trace Formula of Hecke Operators Acting on Newforms

By the Atkin-Lehner theory, the trace formula of Hecke operators acting on newforms can be derived from that on cusp forms. Consider the relation mentioned in Section 3.4:

$$\mathcal{B}_k(N) = \{f(m\tau) : f \in \mathcal{M}_k(\Gamma_1(M))^{\text{new}} \text{ and } mM \mid N\}.$$

If we assume that $(N, n) = 1$, then for any $f \in \mathcal{M}_k(\Gamma_1(M))^{\text{new}}$ and $mM \mid N$, $f(\tau)$ and $f(m\tau)$ have the same T_n-eigenvalues [Diamond and Shurman, 2005, Proposition 5.8.4], so the traces of Hecke operators satisfy

$$\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N))) = \sum_{d \mid N} \sigma_0(N/d)\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(d))^{\text{new}})$$

where $\sigma_0(m)$ denotes the number of positive divisors of m. In [Cohen and Strömberg, 2017, Theorem 13.5.5] we can find the complete relation without assuming $(n, N) = 1$ and $\chi = \mathbb{1}$:

Theorem 4.12. *Let $k \geq 2, N \in \mathbb{Z}^+$, χ a Dirichlet character modulo N of conductor $\mathfrak{f}$ such that $\chi(-1) = (-1)^k$. Denote by $\chi_{\mathfrak{f}}$ the primitive character modulo $\mathfrak{f}$ equivalent to χ. Then*

$$\mathrm{Tr}(T_n, \mathcal{S}_k(N, \chi)) = \sum_{\mathfrak{f} \mid d \mid N} \sigma_0((N/d)_{(0)}) \sum_{\substack{t \mid N/d, t^2 \mid n \\ (t, M) = 1}} \mu(t)\chi_{\mathfrak{f}}(t)t^{k-1}\mathrm{Tr}(T_{n/t^2}, \mathcal{S}_k(d, \chi)^{\text{new}})$$

where the function $m_{(n)} = m/(m, n^\infty)$. □

Inverting this formula we can obtain the trace formula for newforms. Before giving the trace formula, we first define an arithmetic function:

Definition 4.13. Define the multiplicative arithmetic function $\beta(n)$ on prime powers by

$$\beta(p^e) = \begin{cases} 1 & \text{if } e = 0 \text{ or } 2; \\ -2 & \text{if } e = 1; \\ 0 & \text{if } e \geq 3. \end{cases}$$

Then define $\beta_m(n)$ on prime powers by

$$\beta_m(p^e) = \begin{cases} \beta(p^e) & \text{if } p \nmid m; \\ \mu(p^e) & \text{if } p \mid m. \end{cases}$$

Then we can give the trace formula of Hecke operators acting on the space of newforms.

Theorem 4.14. *Keep the same notation and assumptions from Theorem 4.12. Let $N = N_1N_2$ with $(N_1, N_2) = 1$, N_1 square-free and $\mathrm{ord}_p N_2 \geq 2$ for all $p \mid N_2$. Then*

$$\mathrm{Tr}(T_n, \mathcal{S}_k(N,\chi)^{\mathrm{new}}) = \sum_{\mathfrak{f}\mid d\mid N} \sum_{\substack{t\mid(d/\mathfrak{f}, N_1) \\ t^2\mid n}} \chi_{\mathfrak{f}}(t) t^{k-1} \beta_{n/t^2}(N/d) \mathrm{Tr}(T_{n/t^2}, \mathcal{S}_k(d/t, \chi_{\mathfrak{f}})).$$

Proof. See [Cohen and Strömberg, 2017, Theorem 13.5.7]. □

4.4 Trace Formula of Hecke Operators Acting on $\mathcal{S}_k(\Gamma_1(N))$

By the relation mention in Section 3.4:

$$\mathcal{S}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{S}_k(N,\chi),$$

we can sum up the trace formulas on $\mathcal{S}_k(N,\chi)$ over all Dirichlet characters modulo N to get the trace formula of Hecke operators acting on $\mathcal{S}_k(\Gamma_1(N))$. Therefore, Theorem 4.11 gives us the following trace formula.

Theorem 4.15. *Let $N, n \geq 1$ and $k \geq 2$ be integers. Then for $(N, n) = 1$ and n not a square, the trace of the Hecke operator T_n acting on $\mathcal{S}_k(\Gamma_1(N))$ is:*

$$\begin{aligned} \mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_1(N))) = &- \sum_{s} a(s,k,n) \sum_{f\mid t(s,n)} b(s,f,n) \sum_{\chi} c(s,f,N,n,\chi) \\ &+ \delta_{k,2} \sum_{d\mid n,\ (N,d)=1} \frac{n}{d}, \end{aligned}$$

where s runs over integers such that $s^2 - 4n$ is a positive square or is negative, χ ranges over all Dirichlet characters modulo N, d ranges over positive divisors of N, δ is the Kronecker delta function, h_0 is the class number defined in Section 4.1. □

However, the term $\sum_\chi c(s,f,N,n,\chi)$ seems to be complicated to calculate. The effect of summing over Dirichlet characters can be seen more easily if we derive the trace formula from Theorem 4.6.

Theorem 4.16. *Let $N, n \geq 1$ and $k \geq 2$ be integers. Then the trace of the Hecke operator T_n acting on $\mathcal{S}_k(\Gamma_1(N))$ is:*

$$\begin{aligned}\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_1(N))) = &-\frac{\phi(N)}{2}\sum_{s\leq 4n} p_{k-2}(s,n)\sum_f h_0\left(\frac{s^2-4n}{f^2}\right)\bar{B}(N,(N,f),s,n)\\ &-\frac{1}{4}\sum_{a|n}\min(a,n/a)^{k-1}(\bar{\Phi}(N,a,n/a)+(-1)^k\bar{\Phi}(N,-a,-n/a))\\ &+\delta_{k,2}\sum_{d|n,\ (N,d)=1}\frac{n}{d},\end{aligned}$$

where d ranges over positive divisors of n, χ ranges over all Dirichlet characters modulo N, $\bar{B}(N,(N,f),s,n)$ is defined as

$$\bar{B}(N,f,s,n) = \frac{1}{\phi(N)}\sum_\chi B(N,\chi,f,s,n) = \begin{cases}\frac{\phi_1(N)}{\phi_1(N/f)} & \text{if } Nf \mid s-n-1\\ 0 & \text{otherwise,}\end{cases}$$

$\bar{\Phi}(N,a,n/a)$ is defined to be

$$\bar{\Phi}(N,a,d) = \sum_{\substack{N=rt\\ r|(a-1),t|(d-1)}} \phi((r,t))\phi(N/(r,t)),$$

and δ, h_0, ϕ_1 and p_{k-1} are defined the same as in Theorem 4.6, and $\phi(N)$ is the Euler ϕ-function.

Proof. This is [Popa, 2017, Theorem 5]. It can be done by applying the following property of Dirichlet characters [Diamond and Shurman, 2005, Section 4.3]:

$$\sum_\chi \chi(n) = \begin{cases}\phi(N) & \text{if } n=1,\\ 0 & \text{if } n\neq 1,\end{cases}$$

where χ ranges over all Dirichlet character modulo n. □

Chapter 5

Vanishing of Traces of Hecke Operators

In this chapter we focus on the vanishing of the trace of Hecke operators. After introducing the topic, in Section 5.2 we will describe an algorithm found by J. Rouse (Rouse [2006]) to give all levels N such that the trace of the Hecke operator T_n acting on $\mathcal{S}_k(\Gamma_0(N))$ is zero for given $k \geq 2, n \in \mathbb{Z}$. After that, some extensions of this algorithm will also be given and justified.

5.1 Introduction

The study of the trace of Hecke operators has a long history due to the importance of Hecke operators in the research on modular forms, and there are still many open problems in this area.

Consider $\mathrm{Tr}(T_n, \mathcal{S}_k(N, \mathbb{1})) = \mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N)))$, the trace of the Hecke operator T_n acting on $\mathcal{S}_k(\Gamma_0(N))$. In 1947, Lehmer conjectured in Lehmer et al. [1947] that Ramanujan's function $\tau(n)$, which is defined as the Fourier coefficients of the cusp form η^{24}:

$$\sum_{n\geq 1} \tau(n) q^n = \eta^{24} = q \prod_{n\geq 1} (1-q^n)^{24}$$

with $z \in \mathcal{H}$ and $q = e^{2\pi i z}$, is nonzero for any $n \geq 1$. Since $\mathcal{S}_{12}(\Gamma_0(1))$ has dimension 1 (see Example 3.10) and $\tau(n)$ is the T_n-eigenvalue by Theorem 3.19, $\tau(n) = \mathrm{Tr}(T_n, \mathcal{S}_{12}(\Gamma_0(1)))$, this conjecture can be stated as

Conjecture 5.1 (Lehmer's Conjecture)**.**

$$\mathrm{Tr}(T_n, \mathcal{S}_{12}(\Gamma_0(1))) \neq 0$$

for any $n \in \mathbb{Z}^+$*.*

Lehmer's conjecture can be generalised as follows:

Conjecture 5.2 (Generalised Lehmer's Conjecture)**.** *If* $n \geq 1$ *is not a square,* $(N, n) = 1$ *and* $k = 12$ *or* $k \geq 16$ *is even, then*

$$\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N))) \neq 0.$$

The cases when the weight k is odd or equals 4, 6, 8, 10, 14 are excluded since the set of cusp forms at those weights $\mathcal{S}_k(\Gamma_0(1)) = \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$ (see [Zagier, 2008, Section 1.3]). Apart from the nonvanishing cases, the case when the weight $k = 2$ is different, and we have the following conjecture:

Conjecture 5.3. *Suppose that* $n \geq 1$ *is not a square. Then there exists infinitely many* N *coprime to* n *such that*

$$\mathrm{Tr}(T_n, \mathcal{S}_2(\Gamma_0(N))) = 0.$$

Since the object of study $\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N)))$ involves three independent variables: n, N, k, we can fix two of them and focus on the third one. For example, the study of Ramanujan's function $\tau(n)$ can be seen as the study of the traces for fixed $N = 1$ and $k = 12$, and J-P. Serre found that the density of primes p for which $\tau(p) = 0$ is zero [Serre, 1981, Theorem 15].

In this chapter we will study the vanishing of traces of Hecke operators with the level N of the congruence subgroup varying and the other variables fixed. Specifically, we will first introduce an algorithm to find all the levels N of the congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ such that the trace of Hecke operators $\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N))) = 0$ for fixed k, n with $k \geq 2$ and $(N, n) = 1$. Then some other situations will be studied still with N as the only independent variable.

5.2 Algorithm Determining Trace Zero Levels of Hecke Operators on $\mathcal{S}_k(\Gamma_0(N))$

In this section, J. Rouse's algorithm Rouse [2006] to find all zero-levels N of $\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N)))$ for fixed n, k with $k \geq 2$ and $(N, n) = 1$ will be introduced and explained. Consider Hijikata's version of the Eichler-Selberg trace formula Theorem 4.11. We can see that the only dependence on the level N in the trace formula is the constant term $c(s, f, N, n) := c(s, f, N, n, \mathbb{1})$, so the algorithm will center on this term. First of all we define some concepts to better describe it.

Definition 5.4. For $n, N \in \mathbb{Z}^+$ with n not a square and $(n, N) = 1$, let $V(n, N, \chi)$ be the row vector with entries $c(s, f, N, n, \chi)$ for all $s^2 < 4n$ and $f \mid t(s, n)$, together with $c(n/d+d, f, N, n, \chi)$ for all $d \mid n$, $d < \sqrt{n}$ and $f \mid (n/d-d)$. Denote $V(n, N) := V(n, N, \mathbb{1})$.

Notice that for fixed n, the numbers of elements in $V(n, N, \chi)$ are equal for any $N \in \mathbb{Z}^+$.

Example 5.5. *Consider $n = 5$ and $N = 13$. Then*

$$\begin{aligned} V(5, 13, \mathbb{1}) =& [c(0, 1, 13, 5), c(1, 1, 13, 5), c(-1, 1, 13, 5), c(2, 1, 13, 5), c(-2, 1, 13, 5), \\ & c(2, 2, 13, 5), c(-2, 2, 13, 5), c(3, 1, 13, 5), c(-3, 1, 13, 5), c(4, 1, 13, 5), \\ & c(-4, 1, 13, 5), c(6, 1, 13, 5), c(6, 2, 13, 5), c(6, 4, 13, 5)] \\ =& [0, 0, 0, 2, 2, 2, 2, 0, 0, 2, 2, 2, 2, 2]. \end{aligned}$$

Definition 5.6 (Projective Equivalence)**.** We say that $V(n, N_1, \chi)$ and $V(n, N_2, \chi)$ are projectively equivalent if there exists $0 \neq \lambda \in \mathbb{Q}$ such that $V(n, N_1, \chi) = \lambda V(n, N_2, \chi)$.

Remark 5.7. If $n \geq 1$ is not a square, N_1, N_2 are coprime to n, $k > 2$, and $V(n, N_1) = \lambda V(n, N_2)$, then the trace formula indicates that

$$\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N_1))) = \lambda \mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N_2))).$$

Furthermore, $\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N_1))) = 0$ if and only if $\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N_2)))$.

Definition 5.8. For $n \in \mathbb{Z}^+$ not a square and p a prime, let

$$M(n, p) = \max\left\{ \max_{s^2<4n} (\mathrm{ord}_p(s^2 - 4n)), \max_{d \mid n,\ d<\sqrt{n}} (\mathrm{ord}_p(n/d - d)) \right\} + 1$$

if $(n,p) = 1$ and there exists an s such that $p \mid (s^2 - 4n)$ or a $d < \sqrt{n}$ such that $p \mid (n/d - d)$. Otherwise, let $M(n,p) = 0$. Let $M(n) = \prod_p p^{M(n,p)}$.

Then before we give the algorithm, we prove two lemmas 5.9 and 5.10 essential to the effectiveness of the algorithm.

Lemma 5.9. *If p is a prime and $\mathrm{ord}_p(s^2 - 4n) = e$, then for any $j \geq 1$, we have*

$$c(s, f, p^{e+1}, n) = c(s, f, p^{e+j}, n).$$

Proof. We can check out [Hijikata et al., 1989, Lemma 2.5] to see the solution sets defined in Definition 4.10 explicitly.

In this case, we have $c(s, f, p^{e+j}, n) = c(s, f, p^{e+j}, n, p, \mathbb{1})$ and $\nu = \mathrm{ord}_p p^{e+j} = e + j$. Denote $b = \mathrm{ord}_p f$. By the definition of e and f, we always have $\nu \geq \mathrm{ord}_p(s^2 - 4n) - 2b + 1$. When $s^2 - 4n = p^{2a}d^2$ with $(p,d) = 1$ and a a nonnegative integer, check [Hijikata et al., 1989, Lemma 2.5] and we can find that

$$\bar{A}(s, f, N, n, p) = \left\{ \frac{s \pm p^a d}{2} + zp^{\nu+2b-a} : z \in \mathbb{Z}/p^{a-b}\mathbb{Z} \right\}$$
$$\bar{B}(s, f, N, n, p) = \left\{ \frac{s \pm p^a d}{2} + zp^{\nu+2b-a+1} : z \in \mathbb{Z}/p^{a-b-1}\mathbb{Z}. \right\}$$

Here $\bar{B}(s, f, N, n, p) = \emptyset$ if $a = b$.

Also in all other cases we have $\bar{A}(s, f, N, n, p) = \bar{B}(s, f, N, n, p) = \emptyset$.

We can see that in all cases $c(s, f, p^{e+j}, n, p, \mathbb{1}) = |\hat{A}| + |\hat{B}|$ is independent of $\nu = e + j$, so they have the same value for all integers $j \geq 1$. □

While Rouse's proof [Rouse, 2006, Lemma 3.1] is more arithmetic, the proof given here is straightforward by definition of $c(s, f, p^{e+j}, n)$. However, we will see that giving explicit solution sets can make further extension more easily.

Lemma 5.10. *For fixed $N, n \in \mathbb{Z}^+$ with n not a square and $(N,n) = 1$, there exists $N_0 \mid M(n)$ and $\epsilon_s \in \{0,1\}$ such that $V(n,N)$ is projectively equivalent to a vector whose entries are $\epsilon_s c(s, f, N_0, n)$ followed by $c(n/d + d, f, N_0, n)$.*

Proof. This is Rouse's [Rouse, 2006, Lemma 3.2]. Since it is crucial to the algorithm, we will restate the proof below.

First consider any $p_0 \mid N$ such that $(p_0, M(n)) = 1$. Then by Hensel's lemma (Lemma 2.13) or by checking [Hijikata et al., 1989, Lemma 2.5] we have that $c(s, f, N, n, p_0, \mathbb{1}) = 1 + \left(\frac{s^2-4n}{p_0}\right)$ and $c(n/d + d, f, N, n, p_0, \mathbb{1}) = 2$. Then let

$$\epsilon_s = \begin{cases} 0 & \text{if there exists } p_0 \mid N \text{ such that } (p_0, M(n)) = 1 \text{ and } \left(\frac{s^2-4n}{p_0}\right) = -1 \\ 1 & \text{otherwise} \end{cases}$$

Hence, if $\mathrm{ord}_{p_0} N = r$, then $\epsilon_s c(s, f, N, n) = 2\epsilon_s c(s, f, N/p_0^r, n)$ and $c(n/d+d, f, N, n) = 2c(n/d+d, f, N/p_0^r, n)$. Equivalently, $V(n, N)$ is projectively equivalent to the vector with entries $\epsilon_s c(s, f, N/p_0^r, n)$ followed by $c(n/d + d, f, N/p_0^r, n)$.

Next we can assume that if $p \mid N$ then $p \mid M(n)$. If $N \nmid M(n)$, then there exists prime $p \mid N$ such that $\mathrm{ord}_p(N) > \mathrm{ord}_p M(n)$. Denote $r = \mathrm{ord}_p(N) - \mathrm{ord}_p M(n)$. By Lemma 5.9 we have $c(s, f, N, n) = c(s, f, N/p^r, n)$ and $c(n/d + d, f, N, n) = c(n/d + d, f, N/p^r, n)$, so we can assume that $N \mid M(n)$ and finish the proof. □

This Lemma 5.10 enables the establishment of the algorithm.

Algorithm 1 ($\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N)))$ with $k > 2$)**.** For fixed $n \in \mathbb{Z}^+$ with n not a square and for all $N \in \mathbb{Z}^+$ with $(N, n) = 1$,

Step I Determine all choices of $N_0 \mid M(n)$ and ϵ_s such that

$$\begin{aligned} 0 = &\sum_{s^2<4n} a(s, k, n)\epsilon_s \sum_{f|t(s,n)} b(s, f, n)c(s, f, N_0, n) \\ &+ \sum_{s'} a(s', k, n) \sum_{f|t(s',n)} b(s', f, n)c(s', f, N_0, n) \end{aligned}$$

where s' runs over integers such that $s'^2 - 4n$ is a positive square.

Step II For each pair of choices N_0 and ϵ_s, find all levels N that are multiples of N_0 with $(N, M(n)) = N_0$ such that $V(n, N)$ is projectively equivalent to the vector with entries $\epsilon_s c(s, f, N_0, n)$ followed by $c(n/d + d, f, N_0, n)$. Then those levels N are the zero-levels we want.

This can be done in the following way.

a For some s with $c(s, f, N_0, n) = 0$ for all $f \mid t(s, n)$, the choice of ϵ_s is irrelevant.

b For some s with $\epsilon_s = 0$ and $c(s, f, N_0, n) \neq 0$ for some f, $c(s, f, N, n) = 0$ implies that there exists a prime $p \mid N$ such that $(p, M(n)) = 1$ and $\left(\frac{s^2-4n}{p}\right) = -1$.

c For some s with $\epsilon_s = 1$ and $c(s, f, N_0, n) \neq 0$ for some f, we have $c(s, f, N, n) \neq 0$, so $\left(\frac{s^2-4n}{p}\right) = 0$ or 1 for all $p \mid N$ with $(p, M(n)) = 1$.

d Consider prime p such that $M(n, p) > 0$. Since we are looking for N such that $(N, M(n)) = N_0$, we have that $\mathrm{ord}_p N = \mathrm{ord}_p N_0$ or when $\mathrm{ord}_p N_0 = M(n, p)$ then $\mathrm{ord}_p N$ can be any integers more than $M(n, p)$ by Lemma 5.9.

In conclusion, given a pair of choices N_0 and ϵ_s, all N satisfy the requirements can be decomposed as $N = N_0 N_1 N_2$ such that all prime divisors p of N_1 satisfy $p \mid M(n)$ and all prime divisors q of N_2 satisfy $q \nmid M(n)$. Then the set $\Omega(N_0, \epsilon_s) = \{N = N_0 N_1 N_2\}$ of all N we want includes any integer $N = N_0 N_1 N_2$ such that

a Each prime divisor p of N_1 satisfies $\mathrm{ord}_p N_0 = M(n, p)$;

b For all s satisfying $\epsilon_s = 0$, if $c(s, f, N_0, n) \neq 0$ for some f, then N_2 includes a prime divisor such that $\left(\frac{s^2-4n}{p}\right) = -1$;

c For all s satisfying $\epsilon_s = 1$, if $c(s, f, N_0, n) \neq 0$ for some f, then each prime divisor p of N_2 satisfies $\left(\frac{s^2-4n}{p}\right) = 1$.

Next we talk about the algorithm in the case $k = 2$. The difference of this case is that its trace formula has an extra term. As we assume $(N, n) = 1$, denote this term:

$$\sigma_1(n) = \sum_{d \mid n} \frac{n}{d}.$$

Since it does not include the term $c(s, f, N, n)$, Remark 5.7 no longer holds. However, by the following lemma, a similar algorithm is still available.

Lemma 5.11. *Suppose that $n \in \mathbb{Z}^+$ is not a square. Then there exists an integer $m(n)$ such that if $\mathrm{Tr}(T_n, \mathcal{S}_2(\Gamma_0(N))) = 0$ with $(N, n) = 1$, then N has no more than $m(n)$ distinct prime divisors.*

Proof. See [Rouse, 2006, Theorem 1.1]. □

Similar to Algorithm 1, given $n \in \mathbb{Z}^+$ with n not a square, all N such that $\mathrm{Tr}(T_n, \mathcal{S}_2(\Gamma_0(N))) = 0$ with $(N, n) = 1$ can be decomposed as $N = N_0 N_1 N_2$. This decomposition satisfies that $N_0 \mid M(n)$, all prime divisors p of N_1 satisfy $p \mid M(n)$ and all prime divisors q of N_2 satisfy $q \nmid M(n)$.

Since by Lemma 5.11 the number of prime divisors of N has an upper bound $m(n)$, we can list all possibilities of the distribution of the number of prime divisors on N_0, N_1 and N_2. If N_0 has M different prime divisors, then the number ℓ of prime divisors of N_2 satisfies $0 \leq \ell \leq m(n) - M$. Therefore, $c(s, f, N, n) = 2^\ell \epsilon_s c(s, f, N_0, n)$, where $\epsilon_s = 0$ when there is any prime divisor p of N_2 such that $\left(\frac{s^2-4n}{p}\right) = -1$, or otherwise $\epsilon_s = 1$.

Then we have the following algorithm.

Algorithm 2 $(\mathrm{Tr}(T_n, \mathcal{S}_2(\Gamma_0(N))))$**.** For fixed $n \in \mathbb{Z}^+$ with n not a square and for all $N \in \mathbb{Z}^+$ with $(N, n) = 1$,

Step I Determine all choices of $N_0 \mid M(n)$, $0 \leq \ell \leq m(n) - M$ and ϵ_s such that

$$\begin{aligned}\sigma_1(n) = &\sum_{s^2<4n} a(s,k,n) \sum_{f \mid t(s,n)} b(s,f,n) \cdot 2^\ell \epsilon_s c(s,f,N_0,n) \\ &+ \sum_{s'} a(s',k,n) \sum_{f \mid t(s',n)} b(s',f,n) \cdot 2^\ell c(s',f,N_0,n)\end{aligned}$$

where s' runs over integers such that $s'^2 - 4n$ is a positive square, and M is the number of prime divisors of N_0.

Step II For each choice of N_0, ℓ and ϵ_s, find all levels N that are multiples of N_0 with $(N, M(n)) = N_0$ such that $V(n, N)$ is the vector with entries $2^\ell \epsilon_s c(s, f, N_0, n)$ followed by $2^\ell c(n/d + d, f, N_0, n)$. Then those levels N are the zero-levels we want.

This can be done in the following way.

Given any choice N_0 and ϵ_s, all N we want can be decomposed as $N = N_0 N_1 N_2$ such that all prime divisors p of N_1 satisfy $p \mid M(n)$ and all prime divisors q of N_2 satisfy $q \nmid M(n)$. Then the set $\Omega(N_0, \epsilon_s) = \{N = N_0 N_1 N_2\}$ of all N we want includes any integer $N = N_0 N_1 N_2$ such that

a N_2 has ℓ different prime divisors;

b Each prime divisor p of N_1 satisfies $\mathrm{ord}_p N_0 = M(n, p)$;

c For all s satisfying $\epsilon_s = 0$, if $c(s, f, N_0, n) \neq 0$ for some f, then N_2 includes a prime divisor such that $\left(\frac{s^2-4n}{p}\right) = -1$;

d For all s satisfying $\epsilon_s = 1$, if $c(s, f, N_0, n) \neq 0$ for some f, then each prime divisor p of N_2 satisfies $\left(\frac{s^2-4n}{p}\right) = 1$.

5.3 Algorithm for Hecke Operators on $\mathcal{S}_k(N, \chi)$

In this section, we talk about the algorithm to find levels N such that the trace of Hecke operators T_n acting on $\mathcal{S}_k(N, \chi)$ is zero. This is a generalisation of the last section where we focus on $\chi = \mathbb{1}$.

However, since we fix all variables except the level N, for a fixed Dirichlet character χ modulo $\bar{N}$, $\mathcal{S}_k(N, \chi)$ is well defined only when $\bar{N} \mid N$ and χ is interpreted as the Dirichlet character modulo N induced by the given χ modulo $\bar{N}$. We fix a positive integer $\tilde{N}$ and a Dirichlet character $\tilde{\chi}$. We let N vary over all positive multiples of $\tilde{N}$; for each such N we let χ_N denote a Dirichlet character modulo N induced from $\tilde{\chi}$. The next algorithm is used to find all such levels N such that $\mathrm{Tr}(T_n, \mathcal{S}_k(N, \chi)) = 0$ for fixed $k > 2, n \in \mathbb{Z}^+$.

The next lemma is a generalisation of Lemma 5.9. Before giving the lemma, we first define a generalisation of $M(n)$.

Definition 5.12. For $n \in \mathbb{Z}^+$ not a square, a fixed Dirichlet character $\tilde{\chi}$ modulo $\tilde{N}$ and p a prime, let

$$\bar{M}(n, p, \tilde{\chi}) = \max\left\{2\mathrm{ord}_p\tilde{N}, \max_{s^2<4n}(\mathrm{ord}_p(s^2-4n)), \max_{d|n,\ d<\sqrt{n}}(\mathrm{ord}_p(n/d-d))\right\} + 1,$$

if $(n, p) = 1$ and there exists an s such that $p \mid (s^2 - 4n)$, or a $d < \sqrt{n}$ such that $p \mid (n/d - d)$, or $\mathrm{ord}_p\tilde{N} > 0$. Otherwise, let $\bar{M}(n, p, \tilde{\chi}) = 0$. Let $\bar{M}(n, \tilde{\chi}) = \prod_p p^{\bar{M}(n,p,\tilde{\chi})}$

Lemma 5.13. *If p is a prime, then for any $j \geq 0$, we have*

$$c(s, f, p^{\bar{M}(n,p,\tilde{\chi})}, n) = c(s, f, p^{\bar{M}(n,p,\tilde{\chi})+j}, n).$$

Proof. Similar to the proof of Lemma 5.9, we can again check out [Hijikata et al., 1989, Lemma 2.5] to see the explicit solution sets.

In this case, we have $\nu = \mathrm{ord}_p p^{\bar{M}(n,p,\tilde{\chi})+j} = \bar{M}(n, p, \tilde{\chi}) + j$. Denote $b = \mathrm{ord}_p f$. By the definition of $\bar{M}(n, p, \tilde{\chi})$ and f, we always have $\nu \geq \mathrm{ord}_p(s^2 - 4n) - 2b + 1$. When

$s^2 - 4n = p^{2a}d^2$ with $(p,d) = 1$ and a a nonnegative integer, check [Hijikata et al., 1989, Lemma 2.5] and we can find that

$$\bar{A}(s,f,N,n,p) = \left\{\frac{s \pm p^a d}{2} + zp^{\nu+2b-a} : z \in \mathbb{Z}/p^{a-b}\mathbb{Z}\right\}$$
$$\bar{B}'(s,f,N,n,p) = \left\{\frac{s \pm p^a d}{2} + zp^{\nu+2b-a+1} : z \in \mathbb{Z}/p^{a-b-1}\mathbb{Z}.\right\}$$

Here $\bar{B}'(s,f,N,n,p) = \emptyset$ if $a = b$.

Consider $\chi_p(\frac{s \pm p^a d}{2} + zp^{\nu+2b-a})$ where $\chi_N = \prod_{p|N} \chi_p$. Since $\mathrm{ord}_p \tilde{N} \le \bar{M}(n,p,\tilde{\chi}) - a \le \nu + 2b - a$, we have $p^{\mathrm{ord}_p \tilde{N}} \mid p^{\nu+2b-a}$. By the periodicity of the Dirichlet character, we have $\chi_p(\frac{s \pm p^a d}{2} + zp^{\nu+2b-a}) = \chi_p(\frac{s \pm p^a d}{2})$ for any $z \in \mathbb{Z}/p^{a-b}\mathbb{Z}$, which is independent of ν.

Similarly, since $\mathrm{ord}_p \tilde{N} \le \bar{M}(n,p,\chi) - a < \nu + 2b - a + 1$, we have $p^{\mathrm{ord}_p \tilde{N}} \mid p^{\nu+2b-a+1}$. Thus, we have $\chi_p(\frac{s \pm p^a d}{2} + zp^{\nu+2b-a+1}) = \chi_p(\frac{s \pm p^a d}{2})$ for any $z \in \mathbb{Z}/p^{a-b-1}\mathbb{Z}$, which is also independent of ν.

Also in all other cases we have $\bar{A}(s,f,N,n,p) = \bar{B}(s,f,N,n,p) = \emptyset$.

We have seen that in all cases $c(s,f,p^{\bar{M}(n,p,\tilde{\chi})+j},n,\chi_N)$ is independent of $\nu = \bar{M}(n,p,\tilde{\chi}) + j$, so they have the same value for all integers $j \ge 0$. $\square$

For prime $p \mid N$ with $p \nmid \bar{M}(n,\tilde{\chi})$, the restriction of the induced Dirichlet character χ_N of $\tilde{\chi}$ to $\mathbb{Z}/p^{\mathrm{ord}_p N}$ is the trivial character, so the same argument in the proof of Lemma 5.10 applies.

Lemma 5.14. *For fixed $\tilde{N}, n \in \mathbb{Z}^+$ with n not a square, $\tilde{N} \mid N$ and $(N,n) = 1$, there exists $N_0 \mid \bar{M}(n,\tilde{\chi})$ and $\epsilon_s \in \{0,1\}$ such that $V(n,N,\chi_N)$ is projectively equivalent to a vector whose entries are $\epsilon_s c(s,f,N_0,n,\chi_{N_0})$ followed by $c(n/d+d,f,N_0,n,\chi_{N_0})$.*

Proof. First consider any $p_0 \mid N$ such that $(p_0, M(n,\chi)) = 1$. Since the restriction of the induced Dirichlet character of χ to $\mathbb{Z}/p^{\mathrm{ord}_p N}\mathbb{Z}$ is the trivial character, the same argument in the first half of the proof of Lemma 5.10 applies. Therefore we can again let

$$\epsilon_s = \begin{cases} 0 & \text{if there exists } p_0 \mid N \text{ such that } (p_0, M(n)) = 1 \text{ and } \left(\frac{s^2-4n}{p_0}\right) = -1 \\ 1 & \text{otherwise} \end{cases}$$

Hence, if $\mathrm{ord}_{p_0}N = r$, then $\epsilon_s c(s, f, N, n, \chi_N) = 2\epsilon_s c(s, f, N/p_0^r, n, \chi_N)$ and $c(n/d + d, f, N, n, \chi_N) = 2c(n/d + d, f, N/p_0^r, n, \chi_N)$. Equivalently, $V(n, N, \chi_N)$ is projectively equivalent to the vector with entries $\epsilon_s c(s, f, N/p_0^r, n, \chi_N)$ followed by $c(n/d + d, f, N/p_0^r, n, \chi_N)$.

Next we can assume that if $p \mid N$ then $p \mid \bar{M}(n, \tilde{\chi})$. If $N \nmid \bar{M}(n, \tilde{\chi})$, then there exists prime $p \mid N$ such that $\mathrm{ord}_p(N) > \mathrm{ord}_p\bar{M}(n, \tilde{\chi})$. Denote $r = \mathrm{ord}_p(N) - \mathrm{ord}_p\bar{M}(n, \tilde{\chi})$. By Lemma 5.13 we have $c(s, f, N, n, \chi_N) = c(s, f, N/p^r, n, \chi_N)$ and $c(n/d + d, f, N, n, \chi_N) = c(n/d + d, f, N/p^r, n, \chi_N)$, so we can assume that $N \mid \bar{M}(n, \tilde{\chi})$ and finish the proof. □

With these ingredients we can give a generalisation of Algorithm 1.

Algorithm 3 ($\mathrm{Tr}(T_n, \mathcal{S}_k(N, \chi))$ with $k > 2$ or χ not trivial)**.** For fixed $\tilde{N}, n \in \mathbb{Z}^+$ with n not a square and a Dirichlet character $\tilde{\chi}$ modulo $\tilde{N}$, and for all $N \in \mathbb{Z}^+$ with $(N, n) = 1$ and $\tilde{N} \mid N$, if $k > 2$ or $\tilde{\chi}$ not trivial, we have the following algorithm to find all N, multiples of $\tilde{N}$, such that $\mathrm{Tr}(T_n, \mathcal{S}_k(N, \chi_N)) = 0$ where χ_N is induced by $\tilde{\chi}$.

Step I Determine all choices of $N_0 \mid \bar{M}(n, \tilde{\chi})$ and ϵ_s such that

$$\begin{aligned}0 = &\sum_{s^2<4n} a(s, k, n)\epsilon_s \sum_{f \mid t(s,n)} b(s, f, n)c(s, f, N_0, n, \chi_N)\\ &+ \sum_{s'} a(s', k, n) \sum_{f \mid t(s',n)} b(s', f, n)c(s', f, N_0, n, \chi_N)\end{aligned}$$

where s' runs over integers such that $s'^2 - 4n$ is a positive square.

Step II For each pair of choices N_0 and ϵ_s, find all levels N that are multiples of N_0 with $(N, \bar{M}(n, \tilde{\chi})) = N_0$ such that $V(n, N, \chi_N)$ is projectively equivalent to the vector with entries $\epsilon_s c(s, f, N_0, n, \chi_{N_0})$ followed by $c(n/d + d, f, N_0, n, \chi_{N_0})$. Then those levels N are the zero-levels we want.

This can be done in the following way.

Given a pair of choices N_0 and ϵ_s, all N satisfy the requirements can be decomposed as $N = N_0N_1N_2$ such that all prime divisors p of N_1 satisfy $p \mid \bar{M}(n, \tilde{\chi})$ and all prime divisors q of N_2 satisfy $q \nmid \bar{M}(n, \tilde{\chi})$. Then the set $\Omega(N_0, \epsilon_s) = \{N = N_0N_1N_2\}$ of all N we want includes any integer $N = N_0N_1N_2$ such that

a $\bar{N} \mid N$;

b Each prime divisor p of N_1 satisfies $\mathrm{ord}_p N_0 = \bar{M}(n, p, \tilde{\chi})$;

c For all s satisfying $\epsilon_s = 0$, if $c(s, f, N_0, n, \chi) \neq 0$ for some f, then N_2 includes a prime divisor such that $\left(\frac{s^2-4n}{p}\right) = -1$;

d For all s satisfying $\epsilon_s = 1$, if $c(s, f, N_0, n, \chi) \neq 0$ for some f, then each prime divisor p of N_2 satisfies $\left(\frac{s^2-4n}{p}\right) = 1$.

5.4 Algorithm for Hecke Operators on Newforms

Contrary to the case of cusp forms, we will see that the traces of Hecke operators acting on newforms vanish in most cases when N varies. This is not surprising since Lemma 5.9 shows the stability of traces when the order of any prime divisors of N grows large enough, while the spaces of newforms can be understood as a measure of the differences between different levels of spaces of cusp forms. Before we prove the statements above, we first give the trace formula when $(n, N) = 1$, n is not a square and $\chi = \mathbb{1}$.

Corollary 5.15. *Let $k > 2, n, N \in \mathbb{Z}^+$ such that $(n, N) = 1$ and n is not a square. Then*

$$\begin{aligned}\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N))^{\mathrm{new}}) &= \sum_{d|N} \beta(N/d)\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(d))) \\ &= -\sum_{s} a(s, k, n) \sum_{f|t(s,n)} b(s, f, n) \sum_{d|N} \beta(d) c(s, f, N/d, n)\end{aligned}$$

Proof. Apply Theorem 4.14 and Theorem 4.11 directly. □

Then we use the same notations as Section 5.2 to prove the statements at the beginning of this section.

Lemma 5.16. *Suppose that $k > 2$, p is a prime, $(n, N) = 1$, n is not a square and $\mathrm{ord}_p N \geq \max(M(n, p) + 2, 3)$. Then $\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N))^{\mathrm{new}}) = 0$.*

Proof. Denote $\nu := \mathrm{ord}_p N$. By the trace formula Corollary 5.4 and Lemma 5.9, we have the following for given s, f, N, n

$$\begin{aligned}\sum_{d|N} \beta(d)c(s,f,\frac{N}{d},n) &= \sum_{d|(N/p^\nu)} \{\beta(d)c(s,f,\frac{N}{d},n) + \beta(dp)c(s,f,\frac{N}{dp},n) \\ &\qquad + \beta(dp^2)c(s,f,\frac{N}{dp^2},n)\} \\ &= \sum_{d|(N/p^\nu)} (1-2+1)\ \beta(d)c(s,f,\frac{N}{dp^2},n) \\ &= 0.\end{aligned}$$

Therefore,

$$\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N))^{\mathrm{new}}) = -\sum_s a(s,k,n) \sum_{f|t(s,n)} b(s,f,n) \sum_{d|N} \beta(d)c(s,f,N/d,n) = 0.$$

□

We can first give some definitions analogous to those in Section 5.2.

Definition 5.17. We can define the following functions.

$$\begin{aligned}\tilde{M}(n) &= \prod_{p|M(n)} p^{M(n,p)+2} \\ \tilde{c}(s,f,N,n) &= \sum_{d|N} \beta(d)c(s,f,N/d,n)\end{aligned}$$

and $\tilde{V}(n,N)$ is defined to be the row vector with entries $\tilde{c}(s,f,N,n)$ for all $s^2 < 4n$ and $f \mid t(s,n)$, together with $\tilde{c}(n/d+d,f,N,n)$ for all $d \mid n$, $d < \sqrt{n}$ and $f \mid (n/d-d)$

We then have a result similar to Lemma 5.10.

Lemma 5.18. *For fixed $N, n \in \mathbb{Z}^+$ with n not a square and $(N,n) = 1$, there exists $N_0 \mid \tilde{M}(n)$, $\tilde{\epsilon}' \in \{1,0\}$ and $\tilde{\epsilon}_s \in \{-1,0,1\}$ such that $\tilde{V}(n,N)$ is projectively equivalent to a vector whose entries are $\tilde{\epsilon}_s\tilde{c}(s,f,N_0,n)$ followed by $\tilde{\epsilon}'\tilde{c}(n/d+d,f,N_0,n)$ satisfying that if one of $\tilde{\epsilon}_s = 0$ then $\tilde{\epsilon}' = 0$.*

Proof. We can decompose $N = N_0N_1N_2$ such that $N_0 = (N, \tilde{M}(n))$, all prime divisors p of N_1 satisfy $p \mid \tilde{M}(n)$ and all prime divisors q of N_2 satisfy $q \nmid \tilde{M}(n)$.

First, if $N_1 \neq 1$ or N_2 is not cube-free, then Lemma 5.16 gives that the trace $\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N))^{\mathrm{new}}) = 0$.

Next we can assume that the decomposition of N is $N = N_0N_2$ where N_2 is cube-free. Then for any prime $p \mid N_2$ but $p^2 \nmid N_2$:

$$\begin{aligned}
&\sum_{d|N} \beta(d)c(s, f, \frac{N}{d}, n)\\
&= \sum_{d|(N/p)} \{\beta(dp)c(s, f, \frac{N}{dp}, n) + \beta(d)c(s, f, \frac{N}{d}, n)\}\\
&= \sum_{d|(N/p)} \{\beta(d)\beta(p)c(s, f, \frac{N}{dp}, n) + \beta(d)c(s, f, \frac{N}{dp}, n)(1 + \left(\frac{s^2-4n}{p}\right))\}\\
&= \sum_{d|(N/p)} \beta(d)c(s, f, \frac{N}{dp}, n)(\left(\frac{s^2-4n}{p}\right) - 1)
\end{aligned}$$

Thus, in this case $V(n, N)$ is projectively equivalent to the vector with entries $\tilde{\epsilon}_s\tilde{c}(s, f, N/p, n)$ followed by $\tilde{\epsilon}'\tilde{c}(n/d + d, f, N/p, n) = 0$ where $\tilde{\epsilon}_s \in \{1, 0\}$ and $\tilde{\epsilon}' = 0$.

On the other hand, for any prime $p^2 \mid N_2$:

$$\begin{aligned}
&\sum_{d|N} \beta(d)c(s, f, \frac{N}{d}, n)\\
&= \sum_{d|(N/p^2)} \{\beta(dp^2)c(s, f, \frac{N}{dp^2}, n) + \beta(dp)c(s, f, \frac{N}{dp}, n) + \beta(d)c(s, f, \frac{N}{d}, n)\}\\
&= \sum_{d|(N/p^2)} \beta(d)c(s, f, \frac{N}{dp^2}, n)\left\{1 - 2(1 + \left(\frac{s^2-4n}{p}\right)) + (1 + \left(\frac{s^2-4n}{p}\right))\right\}\\
&= \sum_{d|(N/p^2)} \beta(d)c(s, f, \frac{N}{dp^2}, n)(-\left(\frac{s^2-4n}{p}\right))
\end{aligned}$$

Therefore, in this case $V(n, N)$ is projectively equivalent to the vector with entries $\tilde{\epsilon}_s\tilde{c}(s, f, N/p^2, n)$ followed by $\tilde{\epsilon}'\tilde{c}(n/d + d, f, N/p^2, n) = 0$ where $\tilde{\epsilon}_s \in \{1, -1\}$ and $\tilde{\epsilon}' = 1$.

Since $n/d + d$ is a positive square, $\left(\frac{n/d+d}{p}\right) = 1$ for any prime $p \mid N_2$.

In conclusion, denote $P_1 = \{\text{prime } p : \text{ord}_p N_2 = 1\}$ and $P_2 = \{\text{prime } p : \text{ord}_p N_2 = 2\}$, and then $V(n, N)$ is projectively equivalent to the vector with entries $\tilde{\epsilon}_s\tilde{c}(s, f, N_0, n)$

followed by $\tilde{\epsilon}'\tilde{c}(n/d+d,f,N_0,n)$ as follows:

$$\tilde{\epsilon}' = \begin{cases} 0 & \text{if } P_1 \neq \emptyset \\ 1 & \text{if } P_1 = \emptyset \end{cases}$$

$$\tilde{\epsilon}_s = \left(\prod_{p\in P_1} -\frac{1}{2}(\left(\frac{s^2-4n}{p}\right)-1)\right)\left(\prod_{p\in P_2}\left(\frac{s^2-4n}{p}\right)\right) \in \{1,0,-1\}.$$

This finishes the proof. □

Then we can give the algorithm to find vanishing traces of Hecke operators acting on newforms.

Algorithm 4 ($\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N))^{\mathrm{new}})$ with $k>2$)**.** For fixed $n\in\mathbb{Z}^+$ with n not a square and for all $N\in\mathbb{Z}^+$ with $(N,n)=1$, if $k>2$, we have the following algorithm to find all N such that $\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_0(N))^{\mathrm{new}})=0$

Step I Determine all choices of $N_0 \mid \tilde{M}(n)$ $\tilde{\epsilon}' \in \{1,0\}$ and $\tilde{\epsilon}_s \in \{-1,0,1\}$ satisfying that if one of $\tilde{\epsilon}_s=0$ then $\tilde{\epsilon}'=0$. These choices should also satisfy that

$$\begin{aligned} 0 = &\sum_{s^2<4n} \tilde{\epsilon}_s a(s,k,n) \sum_{f\mid t(s,n)} b(s,f,n)c(s,f,N_0,n) \\ &+ \sum_{s'} \tilde{\epsilon}' a(s',k,n) \sum_{f\mid t(s',n)} b(s',f,n)c(s',f,N_0,n) \end{aligned}$$

where s' runs over integers such that s'^2-4n is a positive square.

Step II Given a choice of N_0, $\tilde{\epsilon}_s$ and $\tilde{\epsilon}'$, all N satisfy the requirements can be decomposed as $N=N_0N_1N_2$ such that all prime divisors p of N_1 satisfy $p\mid \tilde{M}(n,\chi)$ and all prime divisors q of N_2 satisfy $q \nmid \tilde{M}(n)$. Again denote $P_1=\{\text{prime } p : \mathrm{ord}_p N_2 = 1\}$ and $P_2=\{\text{prime } p : \mathrm{ord}_p N_2 = 2\}$. Then the set $\tilde{\Omega}(N_0,\tilde{\epsilon}_s,\tilde{\epsilon}')=\{N=N_0N_1N_2\}$ can be defined to be any integer $N=N_0N_2$ with $(N,n)=1$ satisfying all the following criteria:

a N_2 is cube-free;

b $\tilde{\epsilon}'=0$ if and only if $P_1\neq\emptyset$;

c For all s satisfying $\tilde{\epsilon}_s=0$, there exists a prime $p\in P_1$ such that $\left(\frac{s^2-4n}{p}\right)=1$;

d For all s satisfying $\tilde{\epsilon}_s=-1$, P_2 includes an odd number of prime divisors p such that $\left(\frac{s^2-4n}{p}\right)=-1$, and there is no $p\in P_1$ such that $\left(\frac{s^2-4n}{p}\right)=1$;

e For all s satisfying $\tilde{\epsilon}_s = -1$, P_2 includes an even number of prime divisors p such that $\left(\frac{s^2-4n}{p}\right) = -1$, and there is no $p \in P_1$ such that $\left(\frac{s^2-4n}{p}\right) = 1$.

Step III Consequently, the set of all zero-levels N with $(N, n) = 1$ is the union of $\tilde{\Omega}(N_0, \tilde{\epsilon}_s, \tilde{\epsilon}')$ for all choices N_0, $\tilde{\epsilon}_s$ and $\tilde{\epsilon}'$, and all $N = N_0N_1N_2$ with decomposition components $N_1 \neq 1$ or N_2 not cube-free.

5.5 Algorithm for Hecke Operators on $\mathcal{S}_k(\Gamma_1(N))$

Although finding zero-levels of traces of Hecke operators T_n acting on $\mathcal{S}_k(N, \chi)$ requires a novel algorithm, the zero-levels of those on $\mathcal{S}_k(\Gamma_1(N))$ is surprisingly straightforward.

Observe its trace formula Theorem 4.16. We can find that when $k > 2$, there are only finitely many N such that the trace is possibly not vanishing. Specifically, the term $\bar{B}(N, (N, f), s, n)$ can be nonzero only when $N(N, f) \mid (s - n - 1)$, and the term $\bar{\Phi}(N, a, n/a)$ can possibly be nonzero only when $N \leq (a-1)(d-1)$ by the summing conditions. Consequently, we can find all zero-levels by enumeration.

Algorithm 5 ($\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_1(N)))$)**.** For fixed $\bar{N}, n \in \mathbb{Z}^+$ with n not a square and for all $N \in \mathbb{Z}^+$ with $(N, n) = 1$, if $k > 2$, we have the following algorithm to find all N such that $\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_1(N))) = 0$

Step I For any pair of s, f such that $s^2 < 4n$ and $f \mid t(s, n)$, determine all N such that $(N, f)N \mid (s - n - 1)$. Let Ω_1 denote the set of all such N.

Step II For each $a \in \mathbb{Z}$ with $a \mid n$, find all $1 \leq N \leq (a-1)(n/a - 1)$ such that $\bar{\Phi}(N, a, n/a) \neq 0$. Let Ω_2 denote the set of all such N.

Step III Enumerate each $N \in \Omega_1 \cup \Omega_2$, and see whether $\mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_1(N))) \neq 0$. If $\Omega_0 = \{N \in \Omega_1 \cup \Omega_2 : \mathrm{Tr}(T_n, \mathcal{S}_k(\Gamma_1(N))) \neq 0\}$, then the set of all zero-levels is $\{N \in \mathbb{Z}^+ : (N, n) = 1, N \notin \Omega_0\}$.

Chapter 6

Applications of Traces of Hecke Operators

In this chapter, some applications of traces of Hecke operators are introduced in brief. Most proofs will be skipped and further developments can be seen in references.

6.1 Dimension Formulas

The dimension of $\mathcal{S}_k(N, \chi)$ can be calculated by the Riemann-Roch theorem as in [Shimura, 1971, Section 2.6]. Apart from this geometric approach, we can also calculate the dimension by the trace formula using the fact that the Hecke operator T_1 is the identity map on $\mathcal{S}_k(N, \chi)$ and thus we have the following [Knightly and Li, 2006, Proposition 27.1]:

Proposition 6.1. $\dim \mathcal{S}_k(N, \chi) = \mathrm{Tr}(T_1, \mathcal{S}_k(N, \chi))$. □

A dimension formula of $\mathcal{S}_k(N, \chi)$ simplified from the trace formula with $n = 1$ can be found in [Ross, 1992, Corollary 8]. We give an example to illustrate computing the dimension directly from the first version of the Eichler-Selberg formula, and more examples can be found in [Knightly and Li, 2006, Section 27].

Example 6.2. *Consider $\mathcal{S}_{16}(\mathrm{SL}_2(\mathbb{Z}))$. Let $n = 1$, $N = 1$ and $k = 16$, and then by the Eichler-Selberg trace formula we have the following calculation sheet:*

s	$s^2 - 4n$	f	p_{k-2}	h_0	B
0	-4	1	-1	$\frac{1}{2}$	1
± 1	-3	1	0	$\frac{1}{3}$	1

Thus, the terms in Theorem 4.6 can be computed as follows:

$$\begin{aligned}
(4.2.1a) &= -\frac{1}{2}\cdot(-1)\cdot\frac{1}{2} = \frac{1}{4};\\
(4.2.1b) &= \frac{1}{12}\cdot(16-1) = \frac{5}{4};\\
(4.2.1c) &= -\frac{1}{2};\\
(4.2.1d) &= 0.
\end{aligned}$$

Consequently,

$$\dim \mathcal{S}_{16}(\mathrm{SL}_2(\mathbb{Z})) = \mathrm{Tr}(T_1, \mathcal{S}_{16}(\Gamma_0(1))) = \frac{1}{4}+\frac{5}{4}-\frac{1}{2} = 1,$$

consistent with the result calculated by the dimension formula in [Shimura, 1971, Proposition 2.26]:

$$\dim \mathcal{S}_{16}(\mathrm{SL}_2(\mathbb{Z})) = \left\lfloor \frac{k}{12} \right\rfloor = 1.$$

6.2 Computing Hecke Eigenvalues

For any finite-dimensional linear operator T, let $a_1, \cdots, a_n$ be its eigenvalues, and suppose the characteristic polynomial is

$$\prod_{i=1}^{n}(X-a_i) := \sum_{j=0}^{n}(-1)^j \sigma_j X^{n-j}$$

where $\sigma_0 = 1$. By the definition of traces, we have $\sigma_1 = \mathrm{Tr}\, T$. Moreover, if we denote the power sum of eigenvalues $S_m = \sum_{i=1}^{n} a_i^m = \mathrm{Tr}\, T^m$, then the following classic result gives the connection between coefficients of the characteristic polynomial and the traces:

Proposition 6.3 (Newton-Girard Formula). *For any $m \geq 1$,*

$$m\sigma_m = -\sum_{i=0}^{m-1}(-1)^{m-i}\sigma_i S_{m-i}.$$

Proof. A proof can be found in [Knightly and Li, 2006, Proposition 28.1]. □

By this inductive relation and the trace formula of Hecke operators, we can recover the characteristic polynomial of Hecke operators and then the Hecke eigenvalues without giving an explicit matrix expression of the Hecke operators. See [Knightly and Li, 2006, Section 28].

6.3 The Trace Form of Modular Forms

Consider the Fourier expansion of $\mathcal{T}_{k,\chi,N} : \mathcal{H} \to \mathbb{C}$,

$$\mathcal{T}_{k,\chi,N}(\tau) = \sum_{n\geq 1} \mathrm{Tr}(T_n, \mathcal{S}_k(N,\chi))q^n$$

where $k \geq 2, N \in \mathbb{Z}^+$, $q = e^{2\pi i\tau/N}$, χ a Dirichlet character modulo N and $\chi(-1) = (-1)^k$. We call them "trace forms". H. Cohen conjectured in Cohen [1975] that when $N = 4$, $\mathcal{T}_{k,\chi,N}$ is a modular form. M. Mertens proves in Mertens [2014] that when k is even and χ is trivial, it is in $\mathcal{S}_k(\Gamma_0(4))$, while H. Cohen gives the following general statement.

Theorem 6.4. *$\mathcal{T}_{k,\chi,N}$ is defined above, and let $s = \left\lfloor \frac{k}{12}\phi_1(N)\right\rfloor + 1$ where ϕ_1 is defined in Section 4.1. Then:*

(a) $\mathcal{T}_{k,\chi,N} \in \mathcal{S}_k(N,\chi)$;

(b) The trace forms $T_j(\mathcal{T}_{k,\chi,M})$ and $\bar{\imath}_{N/M} \circ T_j(\mathcal{T}_{k,\chi,M})$ for all $M \mid N$ such that $\mathfrak{f}(\chi) \mid M$ and $a \leq j \leq s/(N/M)$ form a generating set of $\mathcal{S}_k(N,\chi)$, where $\bar{\imath}$ is defined in Section 3.3 and $\mathfrak{f}(\chi)$ is the conductor of χ.

Proof. See [Cohen and Strömberg, 2017, Section 13.5.3]. □

Thus, trace forms are not only cusp forms, but they can also form generating sets of their corresponding space of cusp forms.

Chapter 7

Epilogue

7.1 Review

In this thesis, beyond Rouse's algorithm we give the algorithms to find levels N such that the traces of Hecke operators T_n acting on $\mathcal{S}_k(N,\chi)$, $\mathcal{S}_k(\Gamma_0(N))^{\text{new}}$ or $\mathcal{S}_k(\Gamma_1(N))$ vanish for fixed n, k, χ when $(n, N) = 1$ and n is not a square.

In future research, it is promising to find a similar algorithm for the case when the Hecke operator T_n is replaced by $T_n \circ W_\ell$ where W_ℓ is the Atkin-Lehner operator, since $T_n \circ W_\ell$ has a similar trace formula as T_n. See [Cohen and Strömberg, 2017, §6.6] for the definition and properties of Atkin-Lehner operators, and [Popa, 2017, Theorem 4] for its trace formula.

Moreover, it is worth trying to adapt the algorithm to the cases when $(n, N) \neq 1$ or n is a square. However, either case introduces some extra term dependent on N, and therefore it fails to satisfy the requirements to apply our algorithms directly. Algorithm 2 offers an example to deal with the extra term.

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