Notes on Godement's Notes on Jacquet–Langlands Theory

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1 Representations of $\mathrm{GL}_2(F)$ for a p-adic field F

1.1 The setup

We denote[1] by F a non-archimedean locally compact field, such as for instance $\mathbb{Q}_p$ or a finite extension thereof. We let $\mathscr{O}$ denote the ring of integers of F, and $\mathscr{O}^\times$ the group of units of $\mathscr{O}$ (confusingly to the rest of the world, number theorists insist to refer to this as the units of F; we'll try to avoid this). We fix a non-trivial unitary character τ of the additive group $(F,+)$, $\tau\colon F \to \mathbb{S}^1$. We let $\mathfrak{p}$ denote the unique maximal ideal of $\mathscr{O}$, and we fix a generator ϖ of $\mathfrak{p}$.

The Schwartz–Bruhat functions on F are

$$\mathscr{S}(F) = \{\mathbb{C}\text{-valued locally constant functions with compact support on } F\}$$

Let dx be a Haar measure on $(F,+)$, normalised so that the Fourier inversion formula looks like

$$\hat{f}(y) = \int f(x)\hat{\tau}(xy)dx \Rightarrow f(x) = \int \hat{f}(y)\tau(xy)dy, \quad f \in \mathcal{S}(F)$$

There is an absolute value $|-|$ on F satisfying $d(ax) = |a|dx$.

Let $d^\times x$ be a Haar measure on $F^\times$, chosen so that

$$\int_{\mathscr{O}} d^\times x = 1,$$

so that $d^\times x = c|x|^{-1}dx$, for a constant c.

Let $G = \mathrm{GL}_2(F)$, $K = \mathrm{GL}_2(\mathscr{O})$, so K is a maximal compact open subgroup of G. Let $\mathscr{H}$ be the set of complex-valued locally constant functions with compact support on G. It is the Hecke algebra of G, with the convolution product

$$(f * g)(x) = \int_G f(xy^{-1})g(y)d^\times y$$

where $d^\times y$ is a Haar measure on G such that $\int_K d^\times y = 1$.

1.2 Admissible representations

Let $(\pi, V/\mathbb{C})$ be a representation of G. We say π is admissible if

- V can be written as
$$V = \bigoplus_\theta V(\theta),$$
summing over the irreducible representations θ of K, where $V(\theta)$ is the θ-isotypic subrepresentation of V (restricted to K)

- and $\dim V(\theta) < \infty$.

[1] Some of the notation in Godement is (looking for a polite way to say this) "dated." We chose not to follow it to the letter.

This definition works in both archimedean and non-archimedean cases.
In the p-adic case, this is equivalent to:

- π is smooth, that is for any $v \in V$ there is an open subgroup of G that fixes v
- and for any open subgroup H of G, the space of H-fixed vectors V^H is finite-dimensional.

An admissible representation π of G defines a representation π of the Hecke algebra $\mathscr{H}$ via

$$\pi(f)(v) = \int_G f(g)\pi(g)(v)d^\times g$$

(this is actually a finite sum; consider the open stabiliser of v in G).

Such a representation is called an admissible representation of the Hecke algebra. They can be characterised as follows:

- for any $v \in V$ there is $f \in \mathscr{H}$ such that $\pi(f)v = v$
- and, for all $f \in \mathscr{H}$, $\dim \pi(f)V < \infty$.

(This discussion applies for other reductive groups as well.)

1.3 The Kirillov model

We aim for a concrete realisation of the irreducible admissible representations (π, V) of G.

The finite-dimensional case is very simple, so we get it out of the way now.

Proposition 1.1. *If $\dim V < \infty$, then V has dimension 1 and $\pi = \chi \circ \det$, where χ is a continuous character of $F^\times$.*

This is because $\ker \pi$ is an open normal subgroup, and notice that it contains $\mathrm{SL}_2(F)$ (first see that it contains upper triangular and lower triangular elements for small enough off-diagonal entries, then conjugate by elements of the torus to get anything in $\mathrm{SL}_2(F)$).

Let's consider the infinite-dimensional case now. If (π, V) is a representation of G, a Kirillov model of (π, V) is a representation (π', V') such that

(a) π' is isomorphic to π

(b) V' is a subset of the space $(F^\times)^{\mathbb{C}}$ of $\mathbb{C}$-valued functions on $F^\times$

(c) for all $a, b \in F$ and all $x \in F^\times$,

$$\left(\pi'\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} v'\right)(x) = \tau(bx)v'(ax)$$

The big result is:

Theorem 1.2. *Any infinite-dimensional irreducible admissible representation of $G = \mathrm{GL}_2(F)$ has a unique Kirillov model.*

The proof boils down to having a non-vanishing Whittaker functional.

Assume (π', V') is constructed, together with an isomorphism $\varphi: V \to V'$, $v \mapsto v'$, of G-representations. Define a linear functional L on V by $L(v) = v'(1)$. It satisfies:

$$L\left(\pi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} v\right) = \pi'\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} v'(1) = \tau(b)v'(1) = \tau(b)L(v)$$

On the torus, we want

$$L\left(\pi\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}v\right) = \pi'\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}v'(1) = v'(x)$$

So to actually construct π', the plan is to

- construct a Whittaker functional satisfying the above
- set $v'(x) = L\left(\pi\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}v\right)$

Proof of Kirillov's Theorem. (It would be nice to identify where, in this proof, we use:

- *that V is infinite-dimensional*
- *that π is irreducible*
- *that π is admissible*
- *that G is GL_2)*

Let C be a complex vector space. Consider

$$U = \left\{\varphi \in \mathrm{Hom}_{\mathbb{C}}(V, (F^\times)^C) \mid \text{ if } w = \pi\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}v \text{ then } \varphi(w)(x) = \tau(bx)\varphi(v)(ax)\right\}$$

To prove the theorem, it is enough to show, in the case $C = \mathbb{C}$, that

- there exists $\varphi \in U$ that is injective
- $\dim U = 1$

Let

$$V_0 = \left\{v \in V \mid \int_{\mathfrak{p}^{-n}} \overline{\tau(x)}\,\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}v\,dx = 0 \text{ for } n \gg 0\right\}$$

Lemma 1.3. *Given a complex vector space C, there is an isomorphism of vector spaces between U and*

$$\{L \in \mathrm{Hom}_{\mathbb{C}}(V, C) \mid V_0 \subset \ker L\}$$

Now let $C = V/V_0$ and let $L\colon V \to V/V_0$ be the quotient map.
To prove the theorem, it is enough to show

Lemma 1.4. $\dim C = 1$,

(which will, in particular, imply that L is not the zero map), and

Lemma 1.5. *If $\varphi \in U$ corresponds to L under the isomorphism of Lemma 1.3, then φ is injective.*

Godement actually proves Lemma 1.5 first, which seems a bit concerning as C could, a priori, be 0. However, the proof of Lemma 1.5 also shows that $V_0 \subsetneq V$. □

1.3.1 Proof of Lemma 1.3: vector space isomorphism

Given $\varphi \in U$, define $L: V \to C$ by $L(v) = \varphi(v)(1)$.

If $v \in V_0$, then for $n \gg 0$

$$0 = \int_{\mathfrak{p}^{-n}} \overline{\tau(x)}\, L\left(\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) v\, dx = \int_{\mathfrak{p}^{-n}} \overline{\tau(x)}\tau(x)L(v)\, dx = L(v)\int_{\mathfrak{p}^{-n}} dx$$

so $L(v) = 0$.

In the other direction, suppose $L: V \to C$ vanishes on V_0. Define $\varphi: V \to (F^\times)^C$ by

$$\varphi(v)(x) = L\left(\pi\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} v\right)$$

If

$$w = \pi\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix} v,$$

then

$$\varphi(w)(x) = L\left(\pi\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\pi\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix} v\right) = L\left(\pi\begin{pmatrix}1 & bx\\ 0 & 1\end{pmatrix}\pi\begin{pmatrix}ax & 0\\ 0 & 1\end{pmatrix} v\right)$$

We want to show that the latter is equal to

$$\tau(bx)\varphi(v)(ax) = \tau(bx)L\left(\pi\begin{pmatrix}ax & 0\\ 0 & 1\end{pmatrix}\right),$$

which would follow from

$$L\left(\pi\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix} v\right) = \tau(c)L(v)$$

which would in turn follow from: for any $v \in V$ and any $c \in F$,

$$\pi\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix} v - \tau(c)v \in V_0$$

in other words that for $n \gg 0$ we have

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(x)}\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\left(\pi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} v - \tau(b)v\right) dx = 0$$

We rewrite the integral as

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(x)}\pi\begin{pmatrix}1 & b+x\\ 0 & 1\end{pmatrix} v\, dx - \int_{\mathfrak{p}^{-n}} \overline{\tau(x-b)}\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} v\, dx$$

then choose $n \gg 0$ so that $b \in \mathfrak{p}^{-n}$ and substitute $y = b + x$ in the first integral.

So $\varphi \in U$.

It's very easy to see that the two maps $\varphi \mapsto L$ and $L \mapsto \varphi$ defined above are $\mathbb{C}$-linear and mutual inverses, so Lemma 1.3 is proved.

1.3.2 Proof of Lemma 1.5: injectivity of φ

Let $v \in V$ be such that $\varphi(v)$ is the zero function. Since

$$\varphi(v)(t) = L\left(\pi\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} v\right),$$

we conclude that

$$\pi\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} v \in V_0 \qquad \text{for all } t \in F^\times$$

Lemma 1.6. *The unipotent subgroup* $U = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$ *fixes* v.

Proof. See Lemma 3, page 6 of Godement's notes. □

We now show that the conclusion of the Lemma implies $v = 0$.

Let $H = \{g \in G \mid \pi(g)v = v\}$. Since π is admissible, H is an open subgroup of G so it is not contained in the Borel subgroup $B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$. So in the Bruhat decomposition

$$G = B \sqcup BwB,$$

H intersects BwB. As it also includes U, it must contain an element of the form $h = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. But U and h generate $\mathrm{SL}_2(F)$. Letting $W = V^{\mathrm{SL}_2(F)}$ we have a G-invariant subspace of V on which G acts as a commutative group. Since $\dim V > 1$ (as V is infinite-dimensional), $W = 0$, so $v = 0$.

1.3.3 Toward the proof of Lemma 1.4

Let $C = V/V_0$ and consider the space of functions

$$V_* = \mathscr{S}_C(F^\times) = \{f\colon F^\times \to C \mid f \text{ locally constant with compact support}\}$$

From here on, we identify V with its image in $(F^\times)^C$ under the injective map φ.

Lemma 1.7. $V_* \subset V$.

Proof. Note that $V_* = \mathscr{S}(F^\times) \otimes_{\mathbb{C}} C$. Given $c \in C$, we set

$$\mathscr{S}_c(F^\times) = \{f \in \mathscr{S}(F^\times) \mid x \mapsto f(x)c \text{ is in } V\}.$$

It is enough to show that $\mathscr{S}_c(F^\times) = \mathscr{S}(F^\times)$ for all c in a spanning set of C.

For $a \in F^\times$, $b \in F$, let $T_{a,b}\colon \mathscr{S}(F^\times) \to \mathscr{S}(F^\times)$ be the operator given by

$$(T_{a,b}f)(x) = \tau(bx)f(ax) \qquad \text{for all } x \in F^\times$$

and let $\mathscr{T} \subset \operatorname{End}\mathscr{S}(F^\times)$ be the subalgebra generated by all the $T_{a,b}$. Then, for any $c \in C$, $\mathscr{S}_c(F^\times)$ is invariant under $\mathscr{T}$. It suffices to prove that

(a) $\mathscr{S}(F^\times)$ is an irreducible $\mathscr{T}$-module;

(b) $\mathscr{S}_c(F^\times) \neq 0$ for c in a spanning set for C.

For (a), consider a $\mathscr{T}$-invariant subspace $S \subset \mathscr{S}(F^\times)$. Given a character χ of $\mathscr{O}^\times$, let

$$S(\chi) = \{f \in S \mid f(ux) = \chi(u)f(x)\},$$

then (see Lemmas 2.2 and 2.3 in the Appendix):

$$S = \bigoplus_\chi S(\chi)$$

Let's extend χ to $F^\times$ by setting

$$\chi_*(x) = \begin{cases} \chi(x) & \text{if } x \in \mathscr{O}^\times \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathscr{S}(F^\times)$ is generated by $\{x \mapsto \chi_*(ax) \mid \chi \text{ character of } \mathscr{O}^\times, a \in F^\times\}$.

We now show that if $S \neq 0$ then $\chi_* \in S$ for all χ. We cannot have $S(\chi) = S$. Therefore there exists $\psi \neq \chi$ with $S(\psi) \neq 0$. Let $0 \neq g \in S(\psi)$. For any $a \in F^\times$, $b \in F$, and $u \in \mathscr{O}^\times$, $T_{ua,ub}g \in S$, so the following function is in S:

$$f(x) = \int_{\mathscr{O}^\times} \tau(ubx)g(uax)\overline{\chi(u)}d^\times u$$

More precisely, $f \in S(\chi)$; for any $v \in \mathscr{O}^\times$ we have (substituting $u' = vu$)

$$f(vx) = \int_{\mathscr{O}^\times} \tau(vubx)g(vuax)\overline{\chi(u)}d^\times u = \int_{\mathscr{O}^\times} \tau(u'bx)g(u'ax)\overline{\chi(u')}\chi(v)d^\times u' = \chi(v)f(x)$$

Given a character λ of $\mathscr{O}^\times$, consider the Gauss sum[2]

$$\gamma(x,\lambda) = \int_{\mathscr{O}^\times} \tau(ux)\lambda(u)d^\times u$$

If λ is not the trivial character of $\mathscr{O}^\times$, then

$$\gamma(x,\lambda) \neq 0 \quad \Leftrightarrow \quad v_\mathfrak{p}(x) = \mathfrak{f}(\tau) - \mathfrak{f}(\lambda),$$

where $\mathfrak{f}(-)$ is the $\mathfrak{p}$-valuation of the conductor of $-$.

We can now rewrite f as

$$f(x) = g(ax)\int_{\mathscr{O}^\times} \tau(ubx)\psi(u)\overline{\chi(u)}d^\times u = \gamma(bx, \psi\overline{\chi})g(ax)$$

and choose $b \in F^\times$ so that

$$\gamma(bx, \psi\overline{\chi}) \neq 0 \quad \Leftrightarrow \quad x \in \mathscr{O}^\times$$

and a so that $g(ax) \neq 0$ if $x \in \mathscr{O}^\times$.

We conclude that f is a scalar multiple of χ_*, hence the latter is in S, so $S = \mathscr{S}(F^\times)$.

It remains to prove (b), namely that $\mathscr{S}_c(F^\times) \neq 0$ for c in a spanning set for C; for instance, we could take those $c \in C$ with a lift $g \in V(\psi)$ for some character ψ of $\mathscr{O}^\times$. If $\chi \neq \psi$, then V contains the function f defined by

$$f(x) = \int_{\mathscr{O}^\times} \tau(ubx)g(uax)\overline{\chi(u)}d^\times u = \gamma(bx, \psi\overline{\chi})g(ax).$$

Taking $a = 1$ and suitable b, this is a nonzero scalar multiple of the function $x \mapsto \chi_*(x)c$ that belongs to V, hence $\mathscr{S}_c(F^\times) \neq 0$. □

[2] To justify calling γ a Gauss sum, recall that τ is of the form $\tau(x) = \exp(2\pi i\{xy\}_p)$ for some $y \in F$.

Lemma 1.8. *For any $b \in F$ and $f \in V$ we have*

$$\pi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f - f \in V_*$$

Proof. Let's call the expression in the statement g. Since

$$\pi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f(x) = \tau(bx) f(x),$$

we have $g(x) = (\tau(bx) - 1) f(x)$.

If $b = 0$ then $\tau(bx) - 1$ is identically zero. Otherwise, it vanishes on a neighbourhood of zero. □

Consider the matrix

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Lemma 1.9. $V = V_* + \pi(w) V_*$.

Proof. Look at the span of

$$\bigcup_{g \in G} \pi(g) V_*$$

This is a nonzero G-invariant subspace of V, so it must actually equal V (as V is irreducible).

But $\pi(B) V_* = V_*$, so the Bruhat decomposition gives us

$$V = V_* + \sum_{b \in B} \pi(bw) V_*$$

We conclude by noting that any $bw \in Bw$ can be written in the form

$$bw = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} wb', \qquad b' \in B,$$

and that

$$\pi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \pi(w) v - \pi(w) v \in V_*$$

so that

$$\pi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \pi(w) v \in V_* + \pi(w) V_*$$

for all $t \in F$ and all $v \in V_*$. □

A family of operators $C \to C$

Given a character χ of $\mathscr{O}^\times$, $t \in F$ and $a \in C$, define

$$\chi_{t,a}(x) = \begin{cases} \chi(t^{-1}x) a & \text{if } x \in t\mathscr{O}^\times \\ 0 & \text{if } x \notin t\mathscr{O}^\times \end{cases}$$

Write $\chi_a = \chi_{1,a}$, then $\chi_{t,a}(x) = \chi_a(t^{-1}x)$.

These functions generate the vector space V_*, and any $v \in V_*$ can be written as

$$v = \sum_{a \in t\mathscr{O}^\times} \sum_{\chi \text{ on } \mathscr{O}^\times} \chi_{t,a} \qquad \text{with} \qquad a = a(t, \chi) = \int_{\mathscr{O}^\times} v(tu) \overline{\chi(u)} \, du$$

Define linear operators $J_\pi(t,\chi)\colon C \to C$ by

$$J_\pi(t,\chi)(a) = \pi(w)\chi_{t,a}(1) = L(\pi(w)\chi_{t,a})$$

Write $\omega_\pi : F^\times \to \mathbb{C}^\times$ for the central character of π. We have

$$\begin{aligned} J_\pi(t,\chi)(a) &= L\left(\pi(w)\pi\begin{pmatrix}1/t & 0\\ 0 & 1\end{pmatrix}\chi_a\right) \\ &= L\left(\pi\begin{pmatrix}1/t & 0\\ 0 & 1/t\end{pmatrix}\pi\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\pi(w)\chi_a\right) \\ &= \omega_\pi(1/t)\pi(w)\chi_a(t) \end{aligned}$$

Lemma 1.10. *For any $f \in V_*$ we have*

$$(\pi(w)f)(x) = \omega_\pi(x)\sum_\chi \int_{F^\times} J_\pi(xy,\chi)f(y)d^\times y$$

Proof. See page 10 of Godement's notes. □

Lemma 1.11. *The operators $J_\pi(t,\chi)$ commute.*

Proof. See Lemma 5, page 11 of Godement's notes. □

Lemma 1.12. *Let $A\colon C \to C$ be $\mathbb{C}$-linear. If A commutes with $\{J_\pi(t,\chi)\}$, then A is a scalar.*

Proof. The linear operator A on C induces a linear operator $T\colon (F^\times)^C \to (F^\times)^C$ by

$$(Tf)(x) = A(f(x))$$

We claim that V is invariant under T, and that $T|_V$ commutes with π, hence (by Schur's Lemma, which holds for admissible irreducible representations) T is a scalar, so that A is a scalar.

If $f \in V_*$ then $Tf \in V_*$. How about $T\pi(w)f$?

$$\begin{aligned} (T\pi(w)f)(x) &= A\pi(w)f(x) \\ &= A\omega_\pi(x)\sum_\chi \int_{F^\times} J_\pi(xy,\chi)f(y)d^\times y \\ &= \omega_\pi(x)\sum_\chi \int_{F^\times} J_\pi(xy,\chi)A(f(y))d^\times y \\ &= \pi(w)(Tf)(x) \end{aligned}$$

So for $f \in V_*$ we have $T\pi(w)f \in \pi(w)V_* \subset V$.

We also see that T commutes with $\pi(b)$ for any $b \in B$. It remains to check that it commutes with $\pi(w)$. We already know that

$$T\pi(w)f = \pi(w)Tf \qquad \text{for } f \in V_*$$

Now suppose $f = \pi(w)g \in \pi(w)V_*$, we want to show that

$$T\pi(w)\pi(w)g = \pi(w)T\pi(w)g,$$

but since $\pi(w)^2 = \pm 1$ both sides are $\pm Tg$. □

Proof of Lemma 1.4. By Lemma 1.12, all the operators $\{J_\pi(t,\chi)\}$ are scalars, so they commute with all operators $A\colon C \to C$. Applying Lemma 1.12 once more, we see that all operators $A\colon C \to C$ are scalars.

Together with the fact (from the proof of Lemma 1.5) that $C \neq 0$, we conclude that $\dim C = 1$. □

1.3.4 Some (further) properties of the Kirillov model

- $v' \in V'$ is locally constant and vanishes outside a compact subgroup of F
- $\mathscr{S}(F^{\times}) \subset V'$ has finite codimension inside V'

2 Appendix: Some background

2.1 Non-archimedean local fields

We write F to denote a non-archimedean local field (feel free to think $F = \mathbb{Q}_p$).

We denote the ring of integers of F by $\mathscr{O}$, and the maximal ideal of $\mathscr{O}$ by $\mathfrak{p}$. We choose a uniformiser (=generator of $\mathfrak{p}$) denoted ϖ, and we write $k = \mathscr{O}/\mathfrak{p}$ for the residue field, p the characteristic of k and q the cardinality of k.

An *additive character* is a nontrivial unitary character $\tau : F \to \mathbb{S}^1$ of the group $(F, +)$. Then $\tau|_{\mathfrak{p}^d} = 1$ for some $d \in \mathbb{Z}$; if we take the minimal possible such d, the fractional ideal $\mathfrak{p}^d$ is the *conductor* of τ.

For $F = \mathbb{Q}_p$, can for instance take τ to be trivial on $\mathscr{O} = \mathbb{Z}_p$ and

$$\tau(1/p^n) = \exp(2\pi i/p^n) \qquad \text{for } n \geqslant 1$$

This has conductor $\mathscr{O} = \mathfrak{p}^0$.

A nice class of functions on $F^\times$ is the space $\mathscr{S}(F^\times)$ of *Schwartz–Bruhat* functions, i.e. locally constant functions $f : F^\times \to \mathbb{C}$ with compact support. The support of such f has a cover by finitely many disjoint open balls B_i, so that f can be written as a (finite) $\mathbb{C}$-linear combination of the characteristic functions

$$1_{B_i}(x) = \begin{cases} 1 & \text{if } x \in B_i \\ 0 & \text{otherwise} \end{cases}$$

The unit group $\mathscr{O}^\times$ acts on the space $\mathscr{S}(F^\times)$ by

$$(u \cdot f)(x) = f(ux) \qquad \text{for } f \in \mathscr{S}(F^\times), u \in \mathscr{O}^\times, x \in F^\times$$

Given $f \in \mathscr{S}(F^\times)$, we can consider the stabiliser

$$H = \{u \in \mathscr{O}^\times | u \cdot f = f\}$$

This has finite index as a subgroup of $\mathscr{O}^\times$.

A *multiplicative character* is a character $\chi : F^\times \to \mathbb{C}^\times$ of the multiplicative group $F^\times$. Then $\chi|_{1+\mathfrak{p}^d} = 1$ for some $d \in \mathbb{Z}_{\geqslant 0}$; if we take the minimal possible such d, the ideal $\mathfrak{p}^d$ is called the *conductor* of χ. If the conductor is $\mathscr{O}^\times$ then we say the character is *unramified.*

Suppose a finite abelian group G acts on a complex vector space V. Given a character χ of G, consider the subspace

$$V(\chi) = \{f \in V \mid u \cdot f = \chi(u) f \text{ for all } u \in G\}.$$

This is a G-invariant subspace of V, called the *χ-isotypical component* of V.

Lemma 2.1. *We have*

$$V = \bigoplus_{\chi} V(\chi)$$

where the sum ranges over all characters χ of G.

Sketch of proof.

(a) If $\chi \neq \mu$ then $V(\chi) \cap V(\mu) = 0$.

(b) Given χ, define $\pi_\chi : V \to V$ by

$$\pi_\chi = \frac{1}{\#G} \sum_{u \in G} \chi(u)^{-1} u \cdot -$$

(c) $\pi_\chi^2 = \pi_\chi$

(d) $\pi_\chi|_{V(\chi)} = 1$

(e) for all $f \in V$, $\pi_\chi(f) \in V(\chi)$

(f) for all $f \in V$

$$f = \sum_\chi \pi_\chi(f)$$

where the sum ranges over all characters χ of G.

□

We need a version of this for infinite groups G, in a special setting:

Lemma 2.2. *Let G be an abelian group acting on a complex vector space V, with the property that the stabiliser of any $f \in V$ has finite index in G. Then*

$$V = \bigoplus_\chi V(\chi)$$

where the sum ranges over all characters χ of G.

Proof. We mimic the finite case. We still have $V(\chi) \cap V(\mu) = 0$ for $\chi \neq \mu$ (the proof doesn't assume finiteness of G).

So we just need to show that any $f \in V$ can be written in the form

$$f = \sum_\chi f_\chi$$

where the sum ranges over all characters χ of G and $f_\chi \in V(\chi)$, with all but finitely many $f_\chi = 0$.

Let H be the stabiliser of f and let χ be a character of G. If $H \subsetneq \ker \chi$, set $f_\chi = 0$. Otherwise, χ induces a character χ' on the finite group G/H and we set

$$f_\chi = f_{\chi'} = \frac{1}{[G:H]} \sum_{uH \in G/H} \chi'(uH)(uH) \cdot f$$

Proving that $f_\chi \in V(\chi)$ is identical to (e) in the finite case.

Applying the result of (f) in the proof of the finite case to the finite group G/H, we get

$$f = \sum_{\chi'} f_{\chi'}$$

and the sum is finite as it ranges over the characters of the finite group G/H. □

A special case that we use in the discussion of Kirillov models is that of the group $\mathscr{O}^\times$ acting on $\mathscr{S}(F^\times)$:

Lemma 2.3. *If* $f \in \mathscr{S}(F^\times)$ *then the stabiliser*

$$H = \{u \in \mathscr{O}^\times \mid u \cdot f = f\}$$

has finite index in $\mathscr{O}^\times$*.*

Proof. It suffices to show that H contains a subgroup of the form $1 + \mathfrak{p}^n$ for some $n \in \mathbb{Z}_{\geqslant 0}$.

Since any $f \in \mathscr{S}(F^\times)$ is a finite linear combination of characteristic functions of open balls of the form $B = B(a, p^m)$, it suffices to consider the case $f = 1_B$. (As the intersection of the stabilisers of all these characteristic functions is contained in H.)

So we need to show that, given $a \in F^\times$ and $m \in \mathbb{Z}$, any ε in a small enough ball around 0 satisfies

$$|(1+\varepsilon)x - a| < p^m \qquad \Leftrightarrow \qquad |x - a| < p^m$$

Using the strong triangle inequality, it suffices to have $|\varepsilon| < p^{m - \max\{m,r\}}$ where $|a| = p^r$. □