Experimental Mathematics 2020: Lab sheets

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1 Introduction to Mathematica (13 March)

Open a new Mathematica notebook and follow along.

Exercise 1.1. Work through
https://www.wolfram.com/language/fast-introduction-for-math-students/en/entering-input/
Note the differences in syntax between Mathematica lists and Sage (Python, really) lists.
Also note the different behaviour of `Range`.

Exercise 1.2. Work through
https://www.wolfram.com/language/fast-introduction-for-math-students/en/fractions-and-decimals/
What is the 5-th digit after the decimal point of the number π^3?
What is the millionth digit after the decimal point of the number π^3?

Exercise 1.3. Work through
https://www.wolfram.com/language/fast-introduction-for-math-students/en/variables-and-functions/
On that page, click on the link Defining Variables and Functions to see a more clear explanation of the difference between `=` and `:=`.
Use the function definition syntax to define the two recursive sequences a and b that converge to the agM of $\sqrt{2}$ and 1.
Compute the first few terms in each sequence.
How many terms do you need in order to get 100 correct digits of the limit?
How many terms do you need in order to get ten thousand correct digits of the limit?

Exercise 1.4. Work through
https://www.wolfram.com/language/fast-introduction-for-math-students/en/sequences-sums-and-series/
Use syntax from that page to define the recursive sequence giving the number of regions cut by n lines in the plane:

$$R(0) = 1, \qquad R(n) = R(n-1) + n.$$

Compute the first few values. Get Mathematica to tell you a formula for the general term of the sequence.

Exercise 1.5. Work through
https://www.wolfram.com/language/fast-introduction-for-math-students/en/plots-in-2d/
https://www.wolfram.com/language/fast-introduction-for-math-students/en/more-plots-in-2d/
https://www.wolfram.com/language/fast-introduction-for-math-students/en/plots-in-3d/
Search the Mathematica documentation for the function `ListPointPlot3D`.

Consider the recursive formula

$$a_0 = x,$$
$$a_n = \frac{1}{2}\left(a_{n-1}^2 + y^2\right)$$

where x and y are fixed parameters.

Compute the first few values a_n for various x and y.

Use `ListPointPlot3D` to get a scatter plot of these values.

Based on the plot, formulate a conjecture regarding the region in the x-y plane where the recursion converges to a finite limit.

Exercise 1.6. Work through

https://www.wolfram.com/language/fast-introduction-for-math-students/en/matrices-and-linear-algebra/

Define the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix}$$

We want to describe all 3×3 matrices B that commute with A.

Work through

https://www.wolfram.com/language/fast-introduction-for-math-students/en/algebra/

Define the matrix of unknowns

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

and ask Mathematica to solve the equation $AB - BA = 0$.

Did you get some conditions on the coefficients b_{ij}?

Good. Now we push this a bit further. I claim that there exist α, β, γ (depending on B) such that

$$B = \alpha A^2 + \beta A + \gamma I.$$

(In other words, that B must be a quadratic polynomial in A.)

Use Mathematica's equation solving capabilities to find α, β, γ in terms of the b_{ij}'s.

Exercise 1.7. Browse through the Mathematica demonstrations at

https://demonstrations.wolfram.com/topic.html?topic=Experimental+Mathematics

2 Constant recognition (20 March)

2.1 Recognising integers

Exercise 2.1. Consider the real number

$$\alpha = \frac{\pi^{10}}{\zeta(10)}$$

Compute α to default precision and guess what the exac value of α might be.

Increase the precision a few times to see if your guess persists.

Exercise 2.2. Consider the real number

$$\alpha = \left(e^{\pi\sqrt{163}} - 744\right)^{1/3}$$

Compute α to default precision and venture a guess about the value of α.

Increase the working precision and recompute α. Do you need to adjust your guess? If not, maybe increase the precision some more.

Think about what strategies you might employ to check whether what you are seeing is just a numerical glitch.

A good approach to this type of question relies on interval arithmetic.

There is some information about interval arithmetic in Mathematica at

https://reference.wolfram.com/language/tutorial/Numbers.html

For Sage, see

https://doc.sagemath.org/html/en/prep/Quickstarts/NumAnalysis.html

which also has other useful hints for working with real numbers.

2.2 Recognising rational numbers

Exercise 2.3. Compute the real number

$$\alpha = \frac{\zeta(20)}{\pi^{20}}$$

and determine experimentally whether it is a rational number. (Use continued fractions.)

Exercise 2.4. Same as above, with

$$\alpha = \frac{\zeta(3)}{\pi^3}$$

Even today, there is much that we (the human race) do not know about the values of ζ at odd integers.

2.3 Lattices

Exercise 2.5. Write a Sage function `lattice2d` with signature

```
def lattice2d(v1, v2, range1, range2):
```

that returns a list of the points in the lattice with basis {`v1, v2`} whose first coefficient is in the range `range1` and the second is in the range `range2`.

Experiment with your function and the square lattice (basis $(1, 0)$ and $(0, 1)$). Use `list_plot` or `scatter_plot` or `points` to visualise the lattice.

Try a couple more lattices.

Exercise 2.6. Repeat the previous exercise, this time in three-dimensional space.

Exercise 2.7. Compare the plots for `lattice2d` for the lattices given by

- $(1, 0)$ and $(0, 1)$, both ranges `range(-3, 4)`
- $(2, 3)$ and $(3, 5)$, both ranges `range(-30, 31)`

You may want to restrict the viewing window for the second plot:

```
points(lst, xmin=-3, xmax=3, ymin=-3, ymax=3)
```

Exercise 2.8. This does not really involve computation, it's more of a pen and paper thing.

What kind of matrices give a valid change of basis for the vector space $\mathbb{R}^2$?

What kind of matrices give a valid change of basis for a lattice L inside the vector space $\mathbb{R}^2$?

Bibliography